CALCULUS TWO   TEST TWO   SOLUTIONS TO SAMPLE QUESTIONS
Show all calculations and simplify answers.   Page 1 of 5.

1. $$\int \frac{1}{\sqrt{5+4x-x^2}}\,dx = \int \frac{1}{\sqrt{9-4+4x-x^2}}\,dx =$$

$$\int \frac{1}{\sqrt{9-(x-2)^2}}\,dx = \sin^{-1}\left(\frac{x-2}{3}\right) + C$$

2. Solve the differential equation $x^{-4}y' = 5e^{-y}$. Separable. $e^y dy = 5x^4 dx$

$\int e^y dy = \int 5x^4 dx \; e^y = x^5 + C \qquad y = \ln(x^5 + C)$

Valid on any interval with $x \neq 0$ and $x > -C^{1/5}$.

3. Solve the differential equation $xy' - 4y = x^5 \cos x$. Standard form is

$y' - \frac{4}{x}y = x^4 \cos x$. The integrating factor is $e^{-\int 4dx/x} = e^{-4\ln x} = x^{-4}$.

$x^{-4}y' - 4x^{-5}y = \cos x, \quad (x^{-4}y)' = \cos x, \quad x^{-4}y = \sin x + C,$

$y = x^4(\sin x + C)$

4. Find the **focus** of the **parabola** $12x = y^2 - 6y + 1$. $12x = y^2 - 6y + 9 - 8,$

$12x + 8 = (y-3)^2, \quad 12\left(x + \frac{2}{3}\right) = (y-3)^2, \quad 4p = 12, \; p = 3,$

The vertex is $\left(-\frac{2}{3}, 3\right)$. The parabola opens to the right, so the focus is at $\left(2\frac{1}{3}, 3\right)$.

5. A **hyperbola** has **vertices** at $(\pm 24, 0)$, and **asymptotes** $y = \pm\frac{5}{6}x$.

(a) Find the **equation** of the hyperbola in **standard** form. $a = 24, \; \frac{b}{a} = \frac{5}{6},$

so $b = 20$. The equation is $\dfrac{x^2}{24^2} - \dfrac{y^2}{20^2} = 1$

(b) Find the **eccentricity** of the hyperbola. $c = \sqrt{24^2 + 20^2} = 4\sqrt{61},$

$e = c/a = 4\sqrt{61}/24 = \frac{1}{6}\sqrt{61}$

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6. Convert the **polar** equation $r = 8\cos\theta - 4\sin\theta$ to one in **rectangular** coordinates, and **describe** the graph. $r = 0$ if $\theta = \tan^{-1}2$, so multiplying by r will not add a point. $r^2 = 8r\cos\theta - 4r\sin\theta$, $x^2 + y^2 = 8x - 4y$.

$$x^2 - 8x + 16 + y^2 + 4y + 4 = 16 + 4, \quad (x-4)^2 + (y+2)^2 = 20.$$

The graph is a **circle** with center at $(4, -2)$ and radius $2\sqrt{5}$.

7. A curve is defined parametrically by $x = 2\cos t$, $y = 5\sin t$.

(a) **Eliminate** the parameter and **identify** the curve. $\frac{x^2}{2^2} + \frac{y^2}{5^2} = 1$ is an ellipse, with center at the origin, semi-major axis 5, semi-minor axis 2, and foci on the y-axis at $(0, \sqrt{21})$. (b) Find $\frac{dy}{dx}$ as a function of the **parameter**, t.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{5\cos t}{-2\sin t} = -\frac{5}{2}\cot(t)$$

8. (a) Convert $(x, y) = (-2, 2\sqrt{3})$ to **polar** coordinates. Let $r = \sqrt{x^2 + y^2} =$

$\sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$. $\tan\theta = \frac{y}{x} = -\sqrt{3}$. To be in Quadrant II, let $\theta = \pi + \tan^{-1}(-\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. $(r, \theta) = \left(4, \frac{2\pi}{3}\right)$

(b) Convert $(r, \theta) = \left(6, \frac{3\pi}{4}\right)$ to **rectangular** coordinates.

$$x = r\cos\theta = 6\cos\frac{3\pi}{4} = 6\left(-\frac{\sqrt{2}}{2}\right) = -3\sqrt{2}$$

$$y = r\sin\theta = 6\sin\frac{3\pi}{4} = 6\left(\frac{\sqrt{2}}{2}\right) = 3\sqrt{2}. \quad (x, y) = (-3\sqrt{2}, 3\sqrt{2})$$

9. Convert the **polar** equation $r = 4\cos\theta + 6\sin\theta$ to one in **rectangular** coordinates, and state what the graph is. Similar to #6. Multiply by r to get the equation of a **circle.**

$$r^2 = 4r\cos\theta + 6r\sin\theta, \quad x^2 + y^2 = 4x + 6y.$$

$$x^2 - 4x + 4 + y^2 - 6y + 9 = 13, \quad (x-2)^2 + (y-3)^2 = 13.$$

The graph is a **circle** with center at $(2, 3)$ and radius $\sqrt{13}$.

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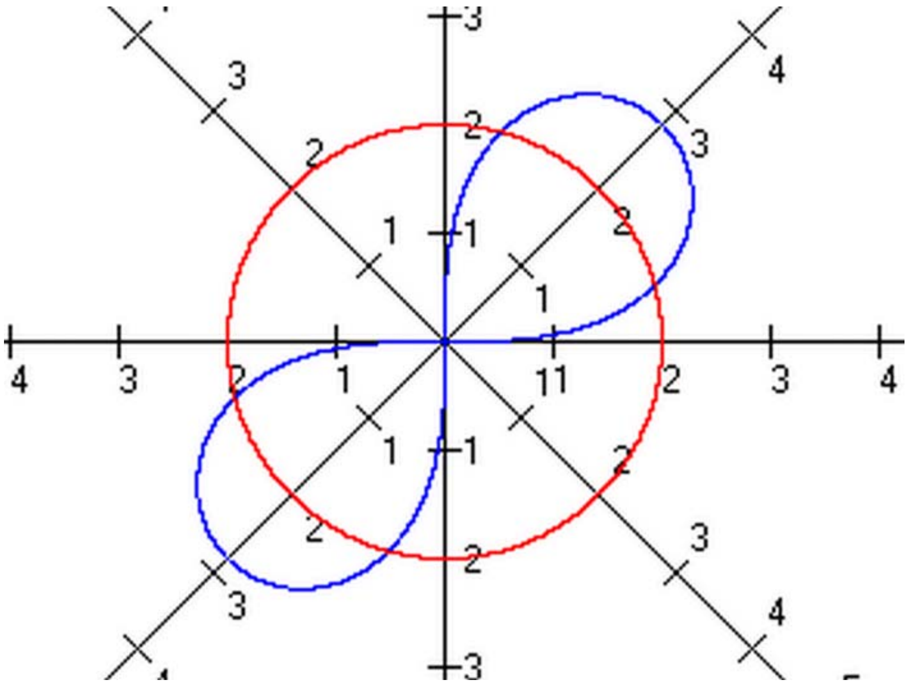
10. Consider the **lemniscate** $r^2 = 8\sin(2\theta)$ and the **circle** $r = 2$.

(a) **Graph** them together, giving **polar coordinates** of the **points of intersection**. (Can use polar graph paper.)



(b) Find the **area of the region outside the circle**, **but inside the lemniscate**.

Substituting into first equation gives $4 = 8\sin(2\theta)$, $\sin(2\theta) = \frac{1}{2}$,

$2\theta = \left\{\sin^{-1}\left(\frac{1}{2}\right),\ \pi - \sin^{-1}\left(\frac{1}{2}\right)\right\} + 2n\pi$, where n is an integer. $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

$\theta = \left\{\frac{\pi}{12},\ \frac{5\pi}{12}\right\} + n\pi$. The desired area is twice the tip of the portion of the lemniscate in the first quadrant, between the points $(r,\ \theta) = \left(2,\ \frac{\pi}{12}\right)$ and $\left(2,\ \frac{5\pi}{12}\right)$.

$$\text{Area} = 2\cdot\frac{1}{2}\int_{\pi/12}^{5\pi/12}(8\sin(2\theta) - 4)d\theta = [-4\cos(2\theta) - 4\theta]_{\pi/12}^{5\pi/12} =$$

$$-4\left(\cos\left(\frac{5\pi}{6}\right) + \frac{5\pi}{12} - \cos\left(\frac{\pi}{6}\right) - \frac{\pi}{12}\right) = -4\left(\frac{-\sqrt{3}}{2} + \frac{4\pi}{12} - \frac{\sqrt{3}}{2}\right) =$$

$$4\left(\sqrt{3} - \frac{\pi}{3}\right).$$

Evaluating this and using the fnInt function on the TI both give 2.739413025.

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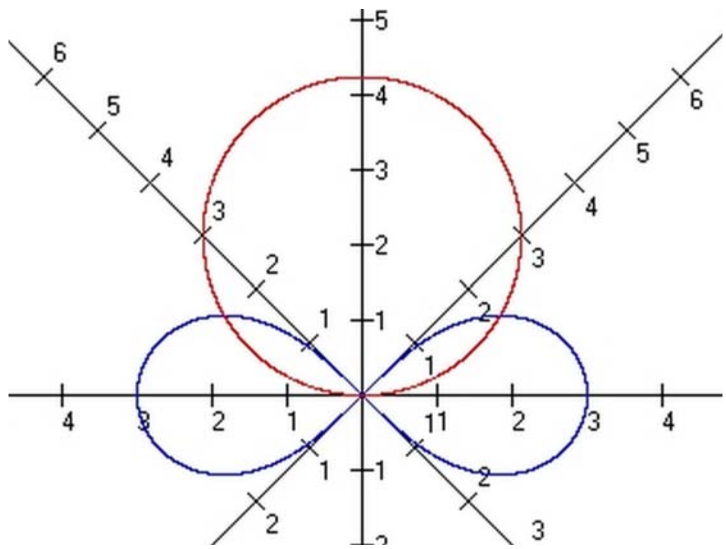
11. (a) Graph the **lemniscate** $r^2 = 9\cos(2\theta)$ and the **circle** $r = 3\sqrt{2}\sin\theta$ together, giving polar coordinates of the **points of intersection**.



(b) **Set up** an integral or integrals to find the **area of the region inside the circle**, **but outside the lemniscate**. Do not compute this area.

Substituting into first equation gives $18\sin^2(\theta) = 9\cos(2\theta)$,

$2\sin^2(\theta) = 1 - 2\sin^2(\theta)$, $\sin^2(\theta) = \frac{1}{4}$, $\sin(\theta) = \pm\frac{1}{2}$. We only need

To use $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ to get $(r, \theta) = \left(\frac{3\sqrt{2}}{2}, \frac{\pi}{6}\right)$ and $\left(\frac{3\sqrt{2}}{2}, \frac{5\pi}{6}\right)$.

By inspection, the pole, $(0, 0)$, is also a point of intersection. The desired area is

twice the right portion. Area = $2 \cdot \frac{1}{2}\left[\int_{\pi/6}^{\pi/2} 18\sin^2(\theta)d\theta - \int_{\pi/6}^{\pi/4} 9\cos(2\theta)d\theta\right] =$

$\int_{\pi/6}^{\pi/2}(9 - 9\cos(2\theta))d\theta - \left[\frac{9}{2}\sin(2\theta)\right]_{\pi/6}^{\pi/4} = \left[9\theta - \frac{9}{2}\sin(2\theta)\right]_{\pi/6}^{\pi/2} - \left[\frac{9}{2}\sin(2\theta)\right]_{\pi/6}^{\pi/4} =$

$3\pi - \frac{9}{2}\sin(\pi) + \frac{9}{2}\sin\left(\frac{\pi}{3}\right) - \frac{9}{2}\sin\left(\frac{\pi}{2}\right) + \frac{9}{2}\sin\left(\frac{\pi}{3}\right) =$

$3\pi + \frac{9}{2}(\sqrt{3} - 1)$.

Evaluating this and using the fnInt function on the TI both give 12.71900659.

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