CALCULUS TWO TEST THREE SAMPLE SOLUTIONS.
Show all calculations and simplify answers. Page 1 of 4.

1. Compute $\sum_{n=1}^{\infty} 3^{(-2n)} = 3^{-2} + 3^{-4} + \ldots = \frac{1}{9} + \frac{1}{81} + \ldots = \frac{a}{1-r} =$

$\frac{1/9}{1-1/9} = \frac{1/9}{8/9} = \frac{1}{8}$. Geometric series with $|r| = \frac{1}{9} < 1$.

2. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+n^2+1}}$ (Apply **limit comparison test** or **comparison test**.) Diverges.

$$a_n = \frac{n}{\sqrt{n^3+n^2+1}} >^{*} \frac{n}{2\sqrt{n^3}} = \frac{1}{2\sqrt{n}}, \quad \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty,$$

since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series with $p = \frac{1}{2} \le 1$. Also $\lim_{n\to\infty} \left|\frac{a_n}{1/\sqrt{n}}\right| =$

$$\lim_{n\to\infty} \left|\frac{n\sqrt{n}}{\sqrt{n^3+n^2+1}}\right| = \lim_{n\to\infty} \left|\frac{1}{\sqrt{1+1/n+1/n^3}}\right| = 1 > 0,$$

so $\sum_{n=1}^{\infty} a_n$ behaves like $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. * True for $n \ge 1$.

3. $\sum_{n=1}^{\infty} n^2 \cdot e^{-n}$ Apply **integral test**. (Integration by parts, or tables.) Converges.

If $y = x^2 e^{-x}$, $y' = x(2-x)e^{-x} < 0$ for $x > 2$ so the (positive) terms are monotonically decreasing for $n \ge 2$.

$$\int_1^{\infty} x^2 e^{-x}\,dx = \left[(-x^2 - 2x - 2)e^{-x}\right]_1^{\infty} = 0 + 5e^{-1} < \infty$$

4. $\sum_{n=1}^{\infty} n! \cdot e^{-n}$ Apply **ratio test**. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| =$

$$\lim_{n\to\infty} \left|\frac{(n+1)!e^{-(n+1)}}{n!e^{-n}}\right| = \lim_{n\to\infty} \left|\frac{n+1}{e}\right| = \infty > 1.$$ Diverges.

5. Classify as **divergent, conditionally convergent** or **absolutely convergent**.
Explanation required.

(a) $\frac{1}{2\sqrt{2}} - \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} - \frac{1}{5\sqrt{5}} + \ldots\ldots = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^{3/2}}$ is absolutely convergent, since $\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series (tail). $p = \frac{3}{2} > 1$.

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5. (b) $$\sum_{n=1}^{\infty} \frac{(-1)^n n\sqrt{n}}{n^2+n} = \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}. \qquad \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$$

If $y = \frac{\sqrt{x}}{x+1}$, $y' = \frac{1-x}{2\sqrt{x}\,(x+1)^2} < 0$ for $x > 1$ so $|a_n|$ is monotonically decreasing to 0 for $n \geq 1$. The series converges by the alternating series test.

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ is a divergent series by limit comparison with $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(See problem number 2.) Conditionaly convergent.

6. Compute by using power series. $\lim_{x\to 0} \frac{1 - \exp(x^2)}{x^2} =$

$$\lim_{x\to 0} \frac{1 - \left(1 + x^2 + (x^2)^2/2! + (x^2)^3/3! + ...\right)}{x^2} = \lim_{x\to 0} \frac{-x^2 - x^4/2 - x^6/6 - .....}{x^2} =$$

$$\lim_{x\to 0} (-1 - x^2/2 - x^4/6 - ....) = -1.$$

7. Find the first four non-zero terms of the Maclaurin series for the following functions.
(Assume removable discontinuities have been removed.)

(a) $$\frac{x^2 - \tan^{-1}(x^2)}{x^4} = \frac{x^2 - (x^2 - x^6/3 + x^{10}/5 - x^{14}/7 + x^{18}/9 - ....)}{x^4} =$$

$$\frac{x^2}{3} - \frac{x^6}{5} + \frac{x^{10}}{7} - \frac{x^{14}}{9} + ...$$

(b) $$(1-2x)^{\frac{5}{2}} = 1 + \tfrac{5}{2}(-2x) + \tfrac{1}{2!}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}(-2x)^2 + \tfrac{1}{3!}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}(-2x)^3 +$$

$$\tfrac{1}{4!}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\cdot(\tfrac{-1}{2})(-2x)^4 + ... = 1 - 5x + \tfrac{15}{2}x^2 - \tfrac{5}{2}x^3 - \tfrac{5}{8}x^4 - .....$$

(c) $y = \sin(e^x - 1)$, $y' = e^x\cos(e^x - 1)$,

$$y'' = e^x\cos(e^x - 1) - e^{2x}\sin(e^x - 1) = y' - e^{2x}y,$$

$$y''' = y'' - 2e^{2x}y - e^{2x}y' = (1 - e^{2x})y' - 3e^{2x}y,$$

$$y^{(4)} = (1 - e^{2x})y'' - 5e^{2x}y' - 6e^{2x}y$$

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7 (c) (Continued.) Only the fifth derivative is needed so y'' was not replaced above.

$y^{(5)} = (1 - e^{2x})y''' - 7e^{2x}y'' - 16e^{2x}y' - 12e^{2x}y$. Evaluating with $x = 0$,

we get $y = 0$, $y' = 1$, $y'' = 1$, $y''' = 0$, $y^{(4)} = -5$, $y^{(5)} = -23$.

$\sin(e^x - 1) = x + \frac{1}{2}x^2 - \frac{5}{24}x^4 - \frac{23}{120}x^5$ .....

8. Suppose a function f has derivatives of all orders, and that
$f(0) = 2$, $f'(0) = 3$, $f''(0) = -4$, $f'''(0) = 0$, $f^{(4)}(0) = 6$, and

$|f^{(5)}(x)| \leq 15$ if $|x| < \frac{1}{2}$.

(a) Find the **4th Maclaurin Polynomial**. $2 + 3x - \frac{4}{2!}x^2 + \frac{0}{3!}x^3 + \frac{6}{4!}x^4$ ..... $=$

$2 + 3x - 2x^2 + \frac{1}{4}x^4$ .....

(b) Find an **upper bound** on the absolute value of the error in using the 4th Maclaurin polynomial for $f(x)$, $|x| < \frac{1}{2}$. $|\text{Error}| \leq \frac{15}{5!}\left(\frac{1}{2}\right)^5 = \frac{1}{256} = 0.00390625$

9. Classify as **divergent, conditionally convergent** or **absolutely convergent**.
Explanation required.

(a) $\frac{1}{\sqrt{2}} - \frac{1}{4} + \frac{1}{\sqrt{3}} - \frac{1}{8} + \frac{1}{\sqrt{4}} - \frac{1}{16} + \frac{1}{\sqrt{5}} - \frac{1}{32} + \ldots\ldots$ diverges to ∞.

The positive terms form the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$, since it is the tail of a p-series with $p = \frac{1}{2} \leq 1$. The negative terms form the series $\sum_{n=2}^{\infty} \frac{-1}{2^n} = -\frac{1}{2}$, since it is a convegent geometric series with $a = \frac{-1}{4}$, $r = \frac{1}{2}$.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n^2+1}$ is absolutely convergent by direct comparison.

$|a_n| = \frac{\sqrt{n}}{n^2+1} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$, for $n \geq 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series with $p = \frac{3}{2} > 1$.

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10. **Estimate** $\int_0^{0.1} \ln(1+x)\,dx$ accurate to **three decimal places** by using Maclaurin series. What test justifies this accuracy?

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \ldots.$$

$$\int_0^{0.1} \ln(1+x)\,dx = \int_0^{0.1} x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \ldots.\,dx =$$

$$\left[\tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{12}x^4 - \tfrac{1}{20}x^5 + \tfrac{1}{30}x^6 - \ldots.\right]_0^{0.1} =$$

$$\tfrac{1}{2}(0.1)^2 - \tfrac{1}{6}(0.1)^3 + \tfrac{1}{12}(0.1)^4 - \tfrac{1}{20}(0.1)^5 + \tfrac{1}{30}(0.1)^6 - \ldots. =$$

$$0.005 - 0.000166666\ldots. + 0.00000833333\ldots. - \ldots.$$

Absolute values of the terms of the alternating series converge to zero monotonically.

Series converges by the alternating series test, with error after n terms less than the absolute value of the next term.

The value of the integral lies between $0.005 - 0.000166666\ldots. = 0.00483333\ldots$

and $0.005 - 0.000166666\ldots. + 0.00000833333\ldots. = 0.0048416666\ldots$

The value is 0.005, to the nearest 0.001. We can actually say it's 0.0048, to the nearest 0.0001, based on the above. One more term allows rounding to 0.00484.

(100 points total.)