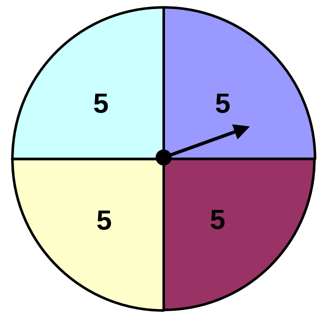
Certain event: An event is absolutely guaranteed to happen every time. The probability of a certain event is one.

Example: Spin the spinner and record the outcome. The event "landing on a 5" is a certain event. P(5) = 1.



Complement of an Event: The complement of an event A is the set of all outcomes in the sample space that are not included in the outcomes of event A.

Notation*:** There are several notations for the complement of event A depending on your resource you might use $\overline{A}$, A^c, or A' and is read "not*** A".

***Rule*:** Given the probability of an event , the probability of its complement can be found by subtracting the given probability from 1. Thus

$$P(A^c) = P(not\ A) = 1 - P(A)$$

Example: Suppose a fair six-sided die is rolled.

Let A = {1, 2}, Then A^c = {3, 4, 5, 6}.

Note that $P(A) = \frac{1}{3}$ and $P(A^c) = \frac{2}{3}$,

so that it is true that $P(A^c) = 1 - P(A)$.

Note1: If two events are complementary, the sum of their probabilities is 1.

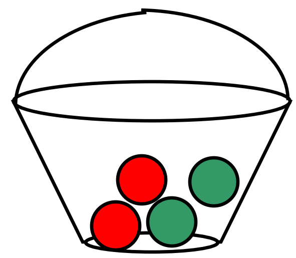
Note 2: Complementary events are mutually exclusive.

<u>Conditional probability</u>: The probability that an event B occurs given that another event A has already occurred.

Notation: $P(B \mid A)$ represents the probability of event *B* occurring after it is assumed that event A has already occurred. (We read $B \mid A$ as "B given A".)

Example: Suppose a bag contains two red balls and two green balls, all of which are identical except for color. One ball is drawn at random and set aside. Then a second ball is drawn at random. Let the events:

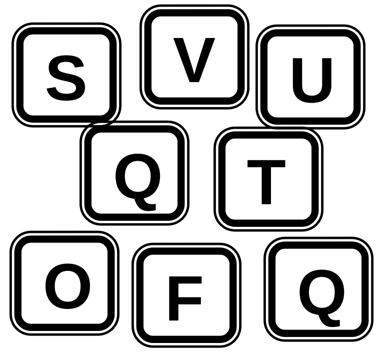
A = the first ball is red
B = the second ball is red



Then the P(A) = 2/4, but the probability that the second ball you pick is red depends on whether the first ball you picked was red or not. You can find the following conditional probabilities

P(the 2^{nd} ball is red given that the 1^{st} ball is red) = P(B|A) = 1/3
P(the 2^{nd} ball is red given that the 1^{st} ball was not red) = P(B|not A) = 2/3

<u>Dependent Events</u>: Events in which the outcome of one event affects the outcome of the other event.



Example: Suppose you are playing a Scrabble® game, and these 8 tiles are left. If you choose two tiles at random, what is the probability you will choose a Q, then another Q?
Since choosing one Q changes the number of tiles and the number of Qs that are left to choose from, the two choices are dependent events. The probability can be found by

$$P(Q, Q) =$$
$$\frac{\textbf{number of Qs}}{\textbf{number of tiles}} = \frac{2}{8} \cdot \frac{1}{7} = \frac{\textbf{number of Qs left after 1st pick}}{\textbf{number of tiles left after 1st pick}}$$
$$= \frac{1}{28}$$

So the probability that you choose a Q, then another Q is $\frac{1}{28}$. 00.0357…, or about 3.5%.

Rule: $P(A\ and\ B) = P(A) \cdot P(B\ given\ A)$

<u>Disjoint Events</u>: Events that cannot occur at the same time. Event A and event B are disjoint if and only if P(A *and* B) = 0. See mutually exclusive events.

<u>Equally likely</u>: Two outcomes or two events are said to be equally likely if they have the same chance or probability of occurring.

Example 1: When tossing a fair coin, the outcome "heads" and outcome "tails" have the same chance of occurring and thus are said to be equally likely.

Example 2: When tossing a fair die, the outcomes 1, 2, 3, 4, 5, & 6 are equally likely since $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$. Also the events "even" and "odd" are also equally likely. Why?

Event: **A**ny subset of the sample space.

Example: Suppose you are asked to pick a number from 1 to 10. Some examples of events might be:

A = {you pick an even number} = {2, 4, 6, 8, 10}
B = {you pick a prime number} = {2, 3, 5, 7}
C = {you pick a three} = { 3 }

Note: Suppose you and your brother both buy some raffle tickets. What is the chance of you or your brother winning? Drawing one of your raffle numbers is a **simple event**. Drawing one of your brother's raffle numbers is another simple event. However, drawing one of your *or* your brother's numbers if a **compound event** because it combines at least two simple events.

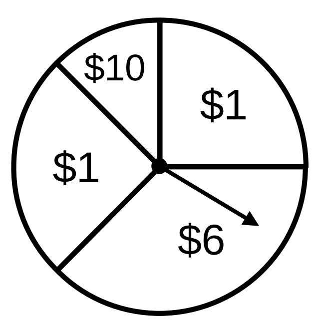
Expected Value: The expected value of an experiment is the average value of the outcomes over many repetitions.

Suppose the outcomes of an experiment are real number (values) called v_1, v_2, …v_n, and that the probabilities of these outcomes are p_1, p_2, … p_n, respectively. Then the expected value, E, of the experiment is the sum
$$E = v_1 \cdot p_1 + v_2 \cdot p_2 + \cdots + v_n \cdot p_n$$

Example: At a carnival if your name is selected you get to spin the spinner shown and win the amount you land on. The expected value of this spinner is

$$\begin{aligned} E &= \$1(\tfrac{1}{2}) + \$6(\tfrac{3}{8}) + \$10(\tfrac{1}{8}) \\ &= \$0.50 + \$2.25 + \$1.25 \\ &= \$4.00 \end{aligned}$$

That means in the long run the average pay out for this spinner would be $4.



<u>Experimental (Empirical) Probability</u>: Experimental probability is the proportion of times a particular outcome actually occurs when a random experiment is repeated a large number of times. See Law of Large Numbers.

$$P(event) = \frac{\text{number of times the event occurs}}{\text{number of trials}}$$

Example: A student flipped a coin 50 times. The coin landed on heads 28 times. The experimental probability of having the coin land on heads is

$$P(\text{heads}) = \frac{28}{50} = 0.56 = 56\%$$

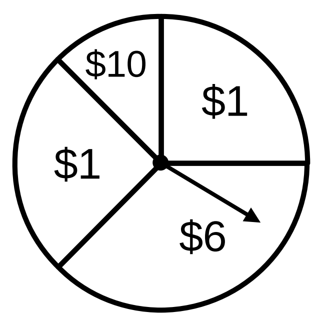
Note: **Experimental Probability can be deceptive**. When a probability experiment has very few attempts or pieces of data, the results can be deceptive. For example, if you rolled a 1—6 number cube three times and rolled a 6 two of the times, the experiment might lead you to say the probability of rolling a 6 would be 2/3. But you know that you don't actually have a 2/3 chance of rolling a 6 every time! This experimental probability of 2/3 would probably drop if you continued to roll the number cube many times.

<u>Fair Game</u>: A game in which each player has an equally likely chance of winning.

Example 1: A game consists of rolling a pair of fair dice. Player A wins if he rolls a sum of 7. Player B wins if she rolls a doubles. This is a fair game since the probability for each player to win is $\frac{1}{6}$.

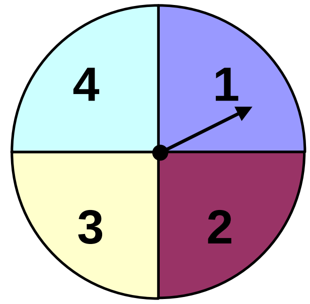
Note: A game is "fair" if and only if the expected value of the game equals the cost of playing the game.

Example 2: In order to make spinning this spinner fair the player would have to pay $4 to play the game.



Impossible Event: An event that cannot happen. The probability of an impossible event is 0.

Example: Spin the spinner and record the outcome. The event "landing on a 5" is an impossible event. P(5) = 0.



Independent Events: Events for which the outcome of one event does not affect the outcome of the other event.

Example1: Tossing a coin and rolling a number cube are independent events.

Rule 1: If two events are independent then given the additional information that one of the events occurred does not influence whether the other event occurs. Hence the conditional probability of event A occurring given that event B occurred is the same as the probability that A occurs.
$P(A|B) = P(A)$

Rule2: If two events are independent then $P(A \text{ and } B) = P(A) \cdot P(B)$.

Example 2: Suppose a school poll shows that 16% of the students play soccer and 75% like pasta. Playing soccer does not affect whether a student likes pasta (and vice versa), so these two events are independent. To find the probability that a randomly selected student plays soccer and likes pasta multiply the respective probabilities.

$$P(\text{plays soccer } and \text{ likes past}) = P(\text{plays soccer}) \cdot P(\text{likes pasta})$$
$$= 0.16 \cdot 0.75$$
$$= 0.12$$

So, the probability that a randomly chosen student in that school plays soccer and likes pasta is 0.12 or 12%.

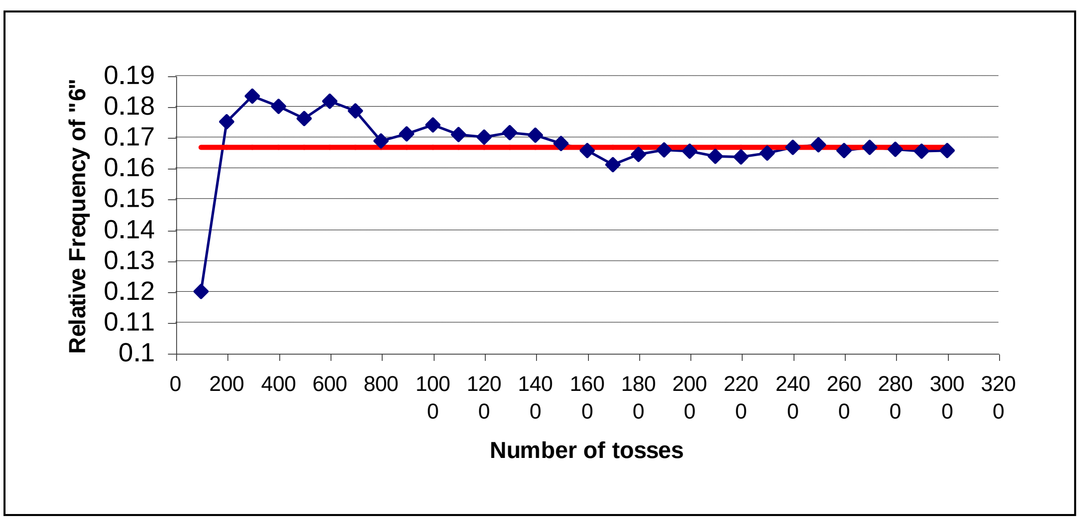
Note: If two events are not independent, then they are dependent. See dependent events.

<u>Law of Large Numbers</u>: The Law of Large Numbers is a fundamental concept in probability that states: If an event of probability *p* observed repeatedly during independent repetitions, the ratio of the observed frequency of that event to the total number of repetitions converges towards *p* as the number of repetitions becomes arbitrarily large.

More simply, as an experiment is repeated over and over, the observed probability approaches the actual (or true) probability.

Example: When a fair die is tossed, the likelihood that the number on the top face of the die will be 2 is $\frac{1}{6}$ because only one of the six numbers on the die is a 2. So the theoretical probability of rolling a two is $\frac{1}{6}$ or about 17%. If a die is tossed six times, a 2 may be rolled more than once or not at all; hence, the percentage of times that a 2 is rolled will vary from the theoretical probability of $\frac{1}{6}$. However, if the die is tossed 600 times, the relative frequency should approximate the theoretical probability. Hence, the number of times the result is 2 after 600 tosses should be fairly close to $600 \times \frac{1}{6}$, or 100.
Suppose each of 100 people rolls a fair die 600 times while keeping track of the percentage of times a 2 was rolled. There most likely would be variations in their resulting relative frequencies. Still, the vast majority of the relative frequencies would be close to $\frac{1}{6}$. If each die were rolled many more times, each of the individual results would tend to be even closer to $\frac{1}{6}$.
(Source: http://www.bookrags.com/Law_of_large_numbers)

This concept can be illustrated by examining the cumulative relative frequency line graph of outcomes for the experiment.



Monte Carlo Simulation: The Monte Carlo method of simulation uses random number devices such as dice, number cubes, coins, spinners, or lists of random numbers (from tables or computers random-number generators) to represent the mathematical characteristics of the real-world situation.

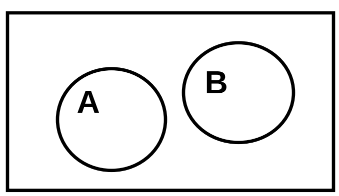
> *Note*: Monte Carlo methods are especially useful for solving problems in which calculating the result theoretically is extremely difficult or beyond the mathematical background of the students.

Mutually Exclusive: Two events that have no outcomes in common are said to be mutually exclusive. Mutually exclusive events cannot occur at the same time, they are disjoint.

> ***Example***: Suppose an experiment consists of rolling a pair of fair dice. The events of **"rolling a sum of 9"** and of **"rolling a double"** have no outcomes in common and thus are mutually exclusive. The events of **"rolling a sum of 6"** and **"rolling a double"** have the outcome (3, 3) in common and thus are NOT mutually exclusive events.

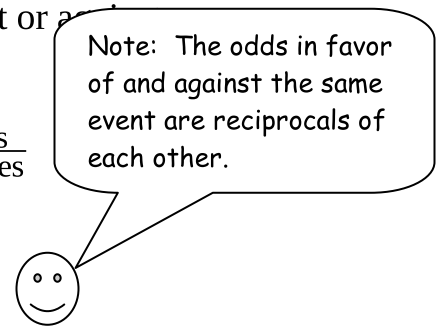
> Notation: If events A and B are mutually exclusive then $A \cap B = \varnothing$

> Venn Diagram:



Events A and B are mutually exclusive or disjoint.

Odds: A ratio involving favorable outcomes and unfavorable outcomes for an event is called odds. Odds can be in favor of an event or a[illegible] event.



$$\text{Odds (in favor of an event)} = \frac{\text{number of favorable outcomes}}{\text{number of unfavorable outcomes}}$$

$$\text{Odds (against an event)} = \frac{\text{number of unfavorable outcomes}}{\text{number of favorable outcomes}}$$

Example 1: The odds in favor of rolling a 3 on a fair six-sided die is 1:5, (read "1 to 5") since there is one outcome that is a 3 on a six-sided die and 5 outcomes that are not 3.

Example 2: The odds against rolling a 3 on a fair six-sided die is 5:1, since there are 5 outcomes that are not 3 and one outcome that is a 3.

Some things to note about odds:

- Odds ***m*:*n*** (read aloud "*m* to *n*") in favor of an event mean we expect the event will occur *m* times for every *n* times it does not occur.
- When the odds in favor of an event are ***m*:*n***, then the probability that the event will occur is $m/\ m+n$
- Odds ***n*:*m*** against an event mean we expect the event will not to occur *n* times for every *m* times it does occur.
- When the odds against an event are ***n*:*m***, then the probability that the event will occur is $m/\ m+n$.
- Odds can be found from the probability that an event occurs by $Odds = probability/\ 1 - probability$
- If you have the odds of an event, the probability is just the odds divided by one plus the odds, i.e. $probability = odds/\ 1 - odds$

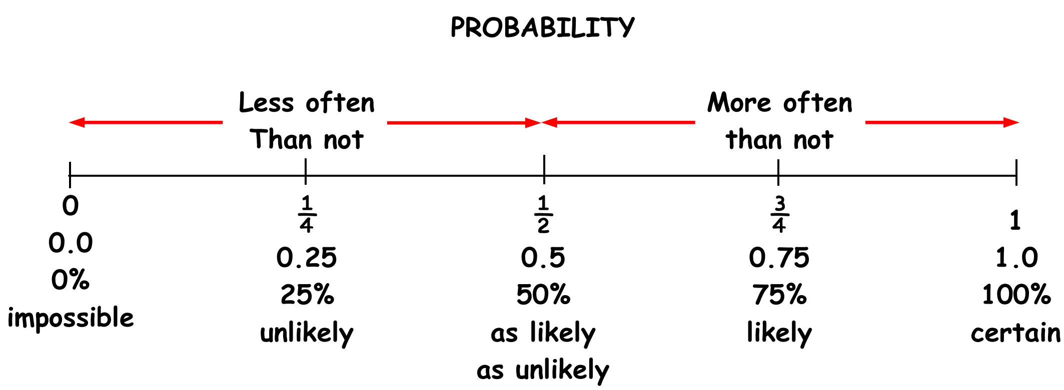
Example 3: It is the bottom of the ninth inning. Any hit or walk will win the game. The batter gets a hit or walk 1 out of every 3 times she is at bat. The odds in favor of winning the game in that time at-bat are ½ , 1:2 or 1 to 2. (For each 3 at-bats there is 1 hit or walk. So, there are 2 at-bats that are not hits or walks.)

Example 4: At a carnival, an average of 3 people in 10 win a prize at a ring toss. The odds *against* winning a prize are $\frac{7}{3}$, 7:3, or 7 to 3. (Out of each 10 people there are 7 who do not win and 3 people who do win.)

<u>Outcome</u>: one of the possible things that can occur as a result of an experiment.

> ***Example 1***: Two coins are tossed. The possible outcomes are H (heads) or T (tails).
>
> ***Example 2***: Two fair dice are rolled and the sum of their faces is recorded. The possible outcomes are {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.

<u>Probability</u>: A number between 0 and 1, inclusive (or between 0% and 100%) that measures how likely it is for a chance event to happen. It can be expressed as a fraction, a decimal, or a percent. At one extreme, events that can't happen have a probability of 0. At the other extreme, events that are certain to happen have a probability of 1. The more *unlikely* an event is, the closer its probability is to 0. The more *likely* an event is, the closer its probability is to 1.



> ***Note 1***: If all the outcomes of an experiment are equally likely, then

$$P(A) = \frac{\text{number of outcomes in event } A}{\text{number of outcomes in the sample space}}$$

Properties of Probability:

I. Basic rules of probability:
- The probability of an event, A, is denoted by $P(A)$.
- For any event A, $0 \le P(A) \le 1$
- The sum of the probabilities of all possible outcomes is 1.
- $P(impossible\ event) = 0$; $P(empty\ set) = P(\varnothing) = 0$
- $P(certain\ event) = 1$; $P(sample\ space) = P(S) = 1$

II. Complement Properties
- $P(not\ A) = 1 - P(A)$
- $P(A) = 1 - P(not\ A)$

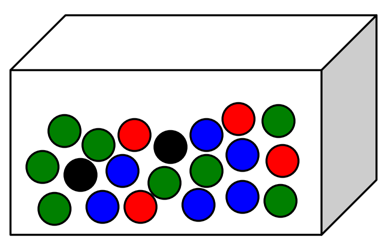
II. Addition Properties of Probability
- "Or" probabilities with mutually exclusive events:
$$P(A\ \mathbf{or}\ B) = P(A) + P(B)$$
- "Or" probabilities with events that are NOT mutually exclusive:
$$P(A\ \mathbf{or}\ B) = P(A) + P(B) - P(A\ and\ B)$$

IV. Multiplication Properties of Probability
- "And" probabilities with dependent events:
$$P(A\ and\ B) = P(A) \cdot P(B\ given\ that\ A\ has\ occurred)$$
- "And" probabilities with independent events:
$$P(A\ and\ B) = P(A) \cdot P(B)$$

Probability Distribution (Probability Table): A table that shows each of the possible outcomes of an experiment paired with its corresponding probability.

Example: A marble is selected at random from a box of 4 blue marbles, 2 black marbles, 8 white marbles and 6 red marbles. The probability distribution for the marble selected is given below.

Outcome	Probability
Blue	$\frac{4}{20} = \frac{1}{5}$ or 20%
Black	$\frac{2}{20} = \frac{1}{10} = 10\%$
White	$\frac{8}{20} = \frac{2}{5} = 40\%$
Red	$\frac{6}{20} = \frac{3}{10} = 30\%$



<u>Sample space</u>: The set of all possible outcomes that can occur as the result of an experiment. Some examples experiments and their corresponding of sample spaces are given below.

> ***Example 1***: Toss a fair coin. S = {heads, tails}
> ***Example 2***: Toss a fair die (six-sided number cube). S = {1, 2, 3, 4, 5, 6}
> ***Example 3***: Toss a fair coin and a fair die.
> S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}

<u>Simulation</u>: Many interesting problem situations are too difficult to analyze mathematically, or they may be inconvenient, impossible, or too dangerous for experimentation. In such circumstances, mathematical models are used to simulate the problem setting and generate data. The data are then analyzed as though they were real data.

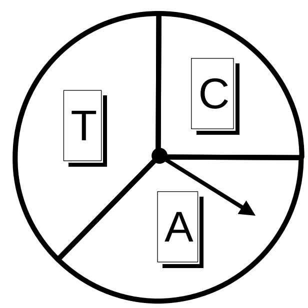
> *Note*: There is a one-to-one correspondence between the outcomes in the original experiment and the outcomes in the simulated experiment. The probability that an outcome in the original experiment occurs is estimated to be the experimental probability of its corresponding outcome in the simulated experiment.

<u>Theoretical probability</u>: The probability obtained by making an assumption of equal likelihood of outcomes in the sample space. Assuming that all outcomes of a sample space have the same chance of occurring then the theoretical probability can be found by the ratio of the number of ways the event can occur to the total number of possibilities in the sample space.

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of ways you can get event E}}{\text{number of possible outcomes}}$$

Example 1: The letters of the word "MATHEMATICS" are placed in a hat and one letters is randomly selected. The probability that the letter selected is a vowel is $\frac{4}{11}$ since there are 4 vowels out of the 11 letters.

Example 2: Theoretical probability may also be based on proportions for example the probability that the spinner shown will land on A is $\frac{3}{8}$ since there are 3 out of 8 equal sections that are A.



<u>Uniform sample space</u>: A sample space in which each outcome has the same chance of occurring. That is all outcomes are equally likely.

Example: When tossing a fair die we are assuming a uniform sample space. That is the sample space {1, 2, 3, 4, 5, 6} consists of six outcomes and each outcome has a probability of $\frac{1}{6}$.