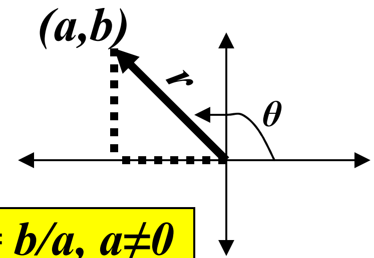
TRIGONOMETRIC IDENTITIES



Let's review the general definitions of trig functions first.
(See back cover of your book)

sin θ = b/r	***cos θ = a/r***	***tan θ = b/a, a≠0***
csc θ = r/b, b≠0	***sec θ = r/a, a≠0***	***cot θ = a/b, b≠0***

By doing some equation manipulating we can establish the **fundamental trigonometric identities**:

tan θ = b/a

Divide both the numerator, b, and the denominator, a, by r and you get

tan θ = (b/r)/(a/r) = sin θ / cos θ

Similarly **cot *θ*= *cos θ / sin θ***

Reciprocal Identities can be established by using the fact that

1/(x/y) = y/x

Therefore:

1/cos *θ* = *sec θ* ***1/sin θ = csc θ*** ***1/tan θ = cot θ***

From the Pythagorean Theorem, we know that $a^2 + b^2 = r^2$

If we divide both sides of that equation by r^2, we get

$$\frac{a^2}{r^2} + \frac{b^2}{r^2} = \frac{r^2}{r^2}$$

$$\left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = 1$$

$$\cos^2\theta + \sin^2\theta = 1$$

Dividing both sides by $\cos^2\theta$, we get:

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta$$

Dividing both sides of $\cos^2\theta + \sin^2\theta = 1$ by $\sin^2\theta$, we get:

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\cot^2\theta + 1 = \csc^2\theta$$

Even-Odd Identities

$\sin(-\theta) = -\sin(\theta)$ $\cos(-\theta) = \cos(\theta)$ $\tan(-\theta) = -\tan(\theta)$

$\csc(-\theta) = -\csc(\theta)$ $\sec(-\theta) = \sec(\theta)$ $\cot(-\theta) = -\cot(\theta)$

Establishing Identities

Combinations of trig functions may be equal to each other, if we can prove it. Once proven, this type of "identity" can be used to reduce an equation down and thus reduce the errors made when calculating values using a computer.

Guidelines for Establishing Identities

1) **It is almost always preferable to start with the side containing the more complicated expression.**
2) **Rewrite sums or differences of quotients as a single quotient.**
3) **Sometimes, rewriting one side in terms of only sines and cosines will help.**
4) **Always keep your goal in mind. As you manipulate one side of the expression, you must keep in mind the form of the expression on the other side.**

In class, we will do p.217 Example 1, Example 3, Example 8

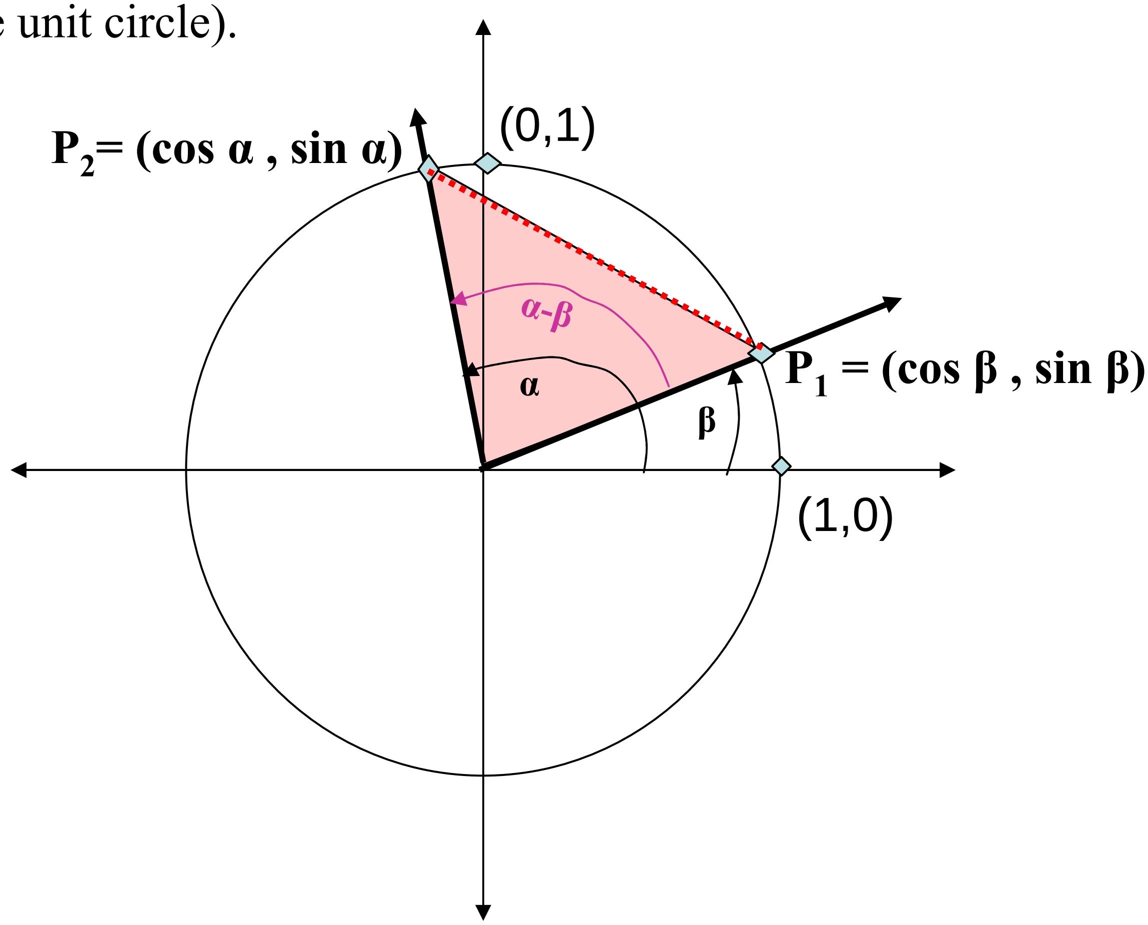
3.4 SUM AND DIFFERENCE FORMULAS

Theorem
cos(α+β) = cos α cos β - sin α sin β
cos(α-β) = cos α cos β + sin α sin β

NOTE:
cos(α+β) ≠ cos α + cos β
cos(α-β) ≠ cos α - cos β

Proof of cos(α-β) = cos α cos β + sin α sin β

Let's use a unit circle so that every point (x,y) on the circle is the cosine and sine of angles in standard position (with the initial side on the positive x-axis and the terminal side with a point somewhere on the unit circle).

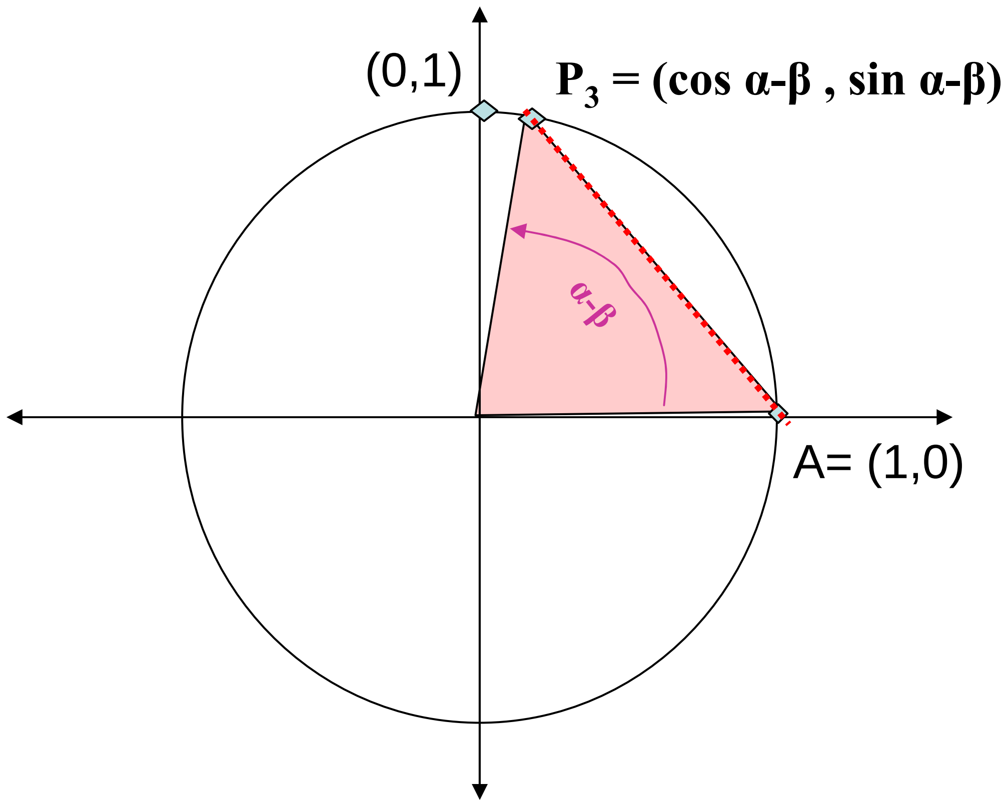


We will use the distance formula to start proving right side of the cos(α-β) equation. The distance from P_1 to P_2 (using the distance formula) is

$$\mathrm{d}(\mathrm{P}_1, \mathrm{P}_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$$

Now to get the left side of the equation, let's rotate that pink triangle so that it has one side on the positive x-axis.



Since this is the same triangle as the previous one (just rotated), then the distance from P_1 to P_2 on the previous triangle is the same as the distance from P_3 to point A on this triangle.

Using the distance formula where the first point is (1,0) and the second point is (cos α-β , sin α-β) we get:

$$d(A,P_3) = \sqrt{(\cos(\alpha-\beta)-1)^2 + (\sin(\alpha-\beta)-0)^2}$$

Setting those two distance equations equal to each other {d(A,P3) = d(P_1,P_2)} we get:

$$\sqrt{(\cos(\alpha-\beta)-1)^2 + (\sin(\alpha-\beta)-0)^2} = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$$

Now we can square both sides:

$$(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2 \quad =(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2$$

Now multiply out the squared terms

$$\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta)=$$

$$\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta$$

Notice with these $\cos^2$ and $\sin^2$ terms we can apply the Pythagorean identity:

$\cos^2\theta + \sin^2\theta = 1$

On the left side, let $\theta = \alpha-\beta$ and we get:

$$1-2\cos(\alpha-\beta)+1$$

$$=2-2\cos(\alpha-\beta)$$

On the right side, let $\theta = \alpha$, and also $\theta = \beta$ and we get:

$$1-2\cos\alpha\cos\beta+1-2\sin\alpha\sin\beta$$

$$=2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta$$

Setting the right side and left side equal to each other we get;

$$2-2\cos(\alpha-\beta)= \quad 2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta$$

Now subtracting 2 from both sides and then dividing both sides by -2 we get;

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

We can now prove

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Use the $\cos(\alpha-\beta)$ formula but instead of $\alpha-\beta$, use $\alpha-(-\beta)$

$\cos(\alpha-(-\beta)) = \cos(\alpha+\beta) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$

Now our even-odd properties [$\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$] will become useful:

$\cos(\alpha+\beta) = \cos\alpha\cos(\beta) - \sin\alpha\sin\beta$

Now if we are asked to find the exact value of a trig function of an Angle, we can perhaps combine two known angles to get an exact answer Remember that your calculator only gives *approximate values.*
Which angles to we know the exact values of their trig functions?
(Look at your handy Unit Circle Diagram)

$\theta = 0°(0° \text{ or } 2\pi_{\text{ rad}}), 30° (\pi/6_{\text{rad}}), 45° (\pi/4_{\text{rad}}), 60° (\pi/3_{\text{rad}}), 90° (\pi/2_{\text{rad}})$

and so on…..

Example 1
Find the exact value of cos 75°

75° is not on our Unit Circle Diagram, but we can use the fact that 75° = 30° + 45°, and we know the trig functions of 30° and 45°.

$\cos(30° + 45°) = \cos 30° \cos 45° - \sin 30° \sin 45°$

Now you do # 5 on p. 231

We can use the Complementary Angle Theorem to find $\sin(\alpha-\beta)$ and $\sin(\alpha+\beta)$

Remember:
$\cos(\pi/2 - \theta) = \sin(\theta)$
$\sin(\pi/2 - \theta) = \cos(\theta)$

Shift the parentheses

$\sin(\alpha+\beta)$ $= \cos(\pi/2 - (\alpha+\beta)) = \cos((\pi/2 - \alpha) + \beta))$

$= \cos(\pi/2 - \alpha) \cos\beta - \sin(\pi/2 - \alpha) \sin\beta$ ← *Use the $\cos(\alpha-\beta)$ formula.*
$= \sin\alpha \cos\beta - \cos\alpha \sin\beta$

Now just use the $\sin(\alpha+\beta)$ for $\sin(\alpha-\beta)$ by using this:

$\sin(\alpha-\beta) = \sin(\alpha + (-\beta)) = \sin\alpha \cos(-\beta) - \cos\alpha \sin(-\beta)$
And using our even-odd properties again, we get:
$\sin(\alpha-\beta)$ $= \sin\alpha \cos\beta - \cos\alpha(-\sin\beta)$
$=$ **$\sin\alpha \cos\beta + \cos\alpha \sin\beta$**

Now let's do Examples 3, 5

Now you do #23 a, b, and c

Formulas for tan(α+β) and tan(α-β)

Proof:

tan(α+β) = $$\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos b-\sin\alpha\sin\beta}$$

Now divide both the numerator and denominator by cos α cos β.

$$\frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$\tan(\alpha+\beta) = \frac{\left(\dfrac{\sin\alpha}{\cos\alpha}\right)+\left(\dfrac{\sin\beta}{\cos\beta}\right)}{1-\left(\dfrac{\sin\alpha}{\cos\alpha}\right)\left(\dfrac{\sin\beta}{\cos\beta}\right)} = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

As before, we can use even-odd properties to derive the other formula..

tan(α-β) = tan(α+(-β)) $$= \frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)}$$

Recall that tan is odd so tan(-θ) = -tan θ, so we get:

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Now do #23d

SUMMARY OF SUM AND DIFFERENCE FORMULAS

cos(α+β) = cos α cos β - sin α sin β
cos(α-β) = cos α cos β + sin α sin β

sin(α+β) = sin α cos β - cos α sin β
sin(α-β) = sin α cosβ + cos α sin β

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Example 9 on p.229

$$\sin\left(\underbrace{\cos^{-1}\frac{1}{2}}_{\alpha}+\underbrace{\sin^{-1}\frac{3}{5}}_{\beta}\right)$$

Remember these are ***angles***

HOMEWORK

p. 220
#1,3, 9, 81
p.231
#5,9,15,23,27,29,59,63