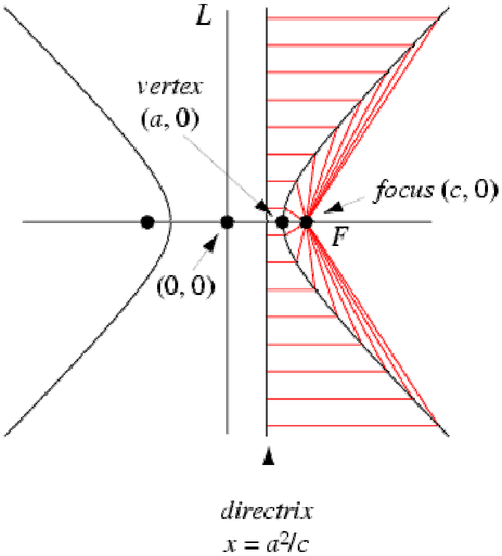
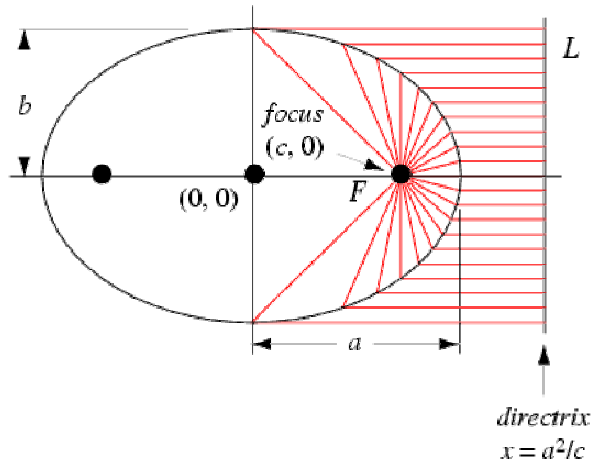
MORE REVIEW OF CONIC SECTIONS

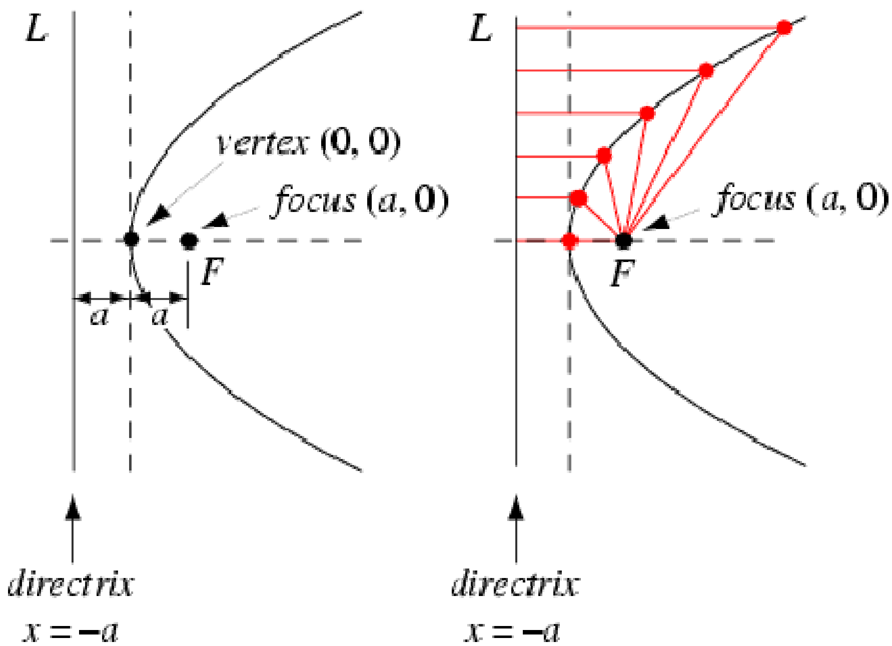
HYPERBOLA



ELLIPSE

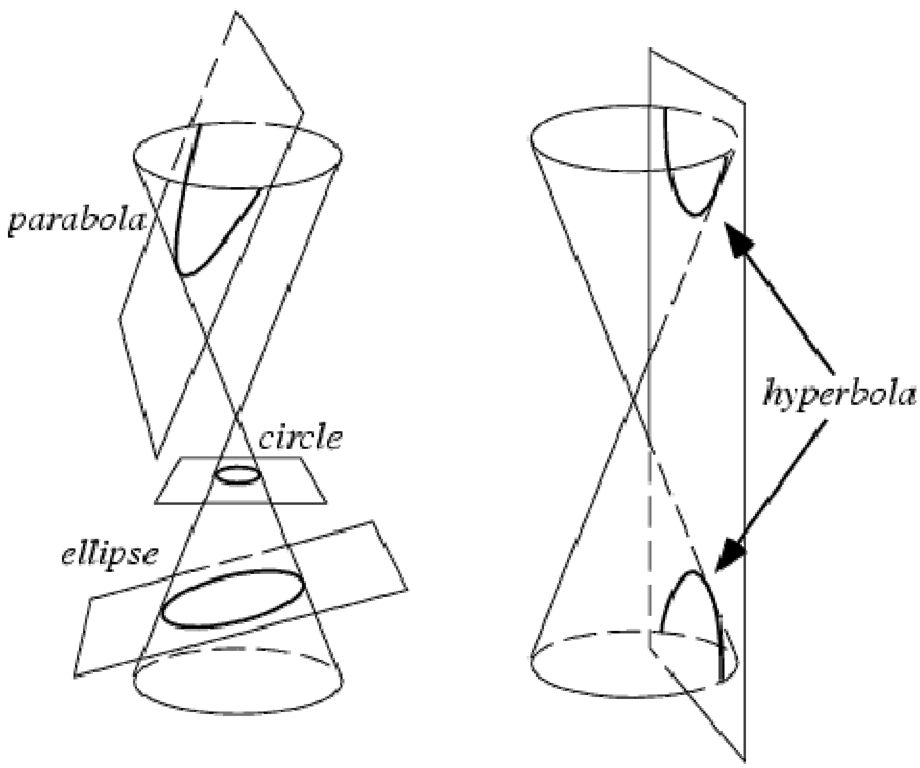


PARABOLA



6.6 POLAR EQUATIONS OF CONICS

A conic section may be defined as the locus of a point **P** that moves in the plane of a fixed point, **F**, called the focus and a fixed line **D** called the conic section directrix (with **F** not on **D**) such that the ratio of the distance P from to F to the distance from F to D is a constant, ***e***, called the eccentricity. If ***e=0*** , the conic is a circle, if **0**< ***e*** < ***1*** , the conic is an ellipse, if ***e=1*** , the conic is a parabola, and if ***e*** > ***1*** , it is a hyperbola. For both the ellipse and the hyperbola, ***e=c/a.***



The distance , p, from the focus to the conic section directrix of a conic section is called the focal parameter. The polar equation of a conic section with focal parameter, p, and eccentricity, e, is given by

Horizontal conics

$$r = \frac{ep}{1 - e\cos q}$$

Directrix is perpendicular to the polar axis at a distance p units to the left of the pole.

$$r = \frac{ep}{1 + e\cos q}$$

Directrix is perpendicular to the polar axis at a distance p units to the right of the pole.

Vertical conics

$$r = \frac{ep}{1 + e\sin q}$$

Directrix is parallel to the polar axis at a distance p units above the pole.

$$r = \frac{ep}{1 - e\sin q}$$

Directrix is parallel to the polar axis at a distance p units below the pole.

Example 3 Discuss and graph this equation, then convert it to a rectangular equation.

$$r = \frac{3}{1+3\cos q}$$

This is the form $r = \dfrac{ep}{1+e\sin q}$ where e = 3, p=1. Therefore the equation is a **hyperbola** since e >0. The directrix is perpendicular to the polar axis (*positive x-axis*) at a distance p=1 unit to the right of the pole *(origin)*. The transverse axis is along the polar axis. Because of this, we know the vertices will occur at θ=0 and θ=π. What does r equal at those points?

Vertex at θ=0 , $r = \dfrac{3}{1+3\cos 0} = \dfrac{3}{1+3(1)} = \dfrac{3}{4}$

Vertex at θ=π , $r = \dfrac{3}{1+3\cos p} = \dfrac{3}{1+3(-1)} = \dfrac{3}{-2}$

The center will be the midpoint of the vertices. To find the rectangular equation for this hyperbola we need to convert everything to rectangular coordinates, and then find a,b,c, and (h,k).

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,\ \text{where}\ c^2 = a^2 + b^2$$

where a is the distance from the center to a vertex.

First, convert the vertices to rectangular coordinates. Remember x=rcosθ, y=rsinθ

Vertex (¾,0): x=(¾)cos0 = ¾, y =(¾)sin0 = 0   (¾ , 0)

Vertex (-3/2,π): x=(-3/2)cosπ = 3/2, y=(-3/2)sin0 = 0 (3/2 , 0)

Center (h,k) = $\left(\dfrac{3/4 + 3/2}{2}, \dfrac{0+0}{2}\right) = \left(\dfrac{3/4 + 6/4}{2}, 0\right) = \left(\dfrac{9/4}{2}, 0\right) = \left(\dfrac{9}{8}, 0\right)$

Distance from center to vertex =

$a = \sqrt{\left(3/4 - (9/8)\right)^2 + (0-0)^2} = \sqrt{(3/8)^2} = 3/8$

$b = \sqrt{c^2 - a^2} = \sqrt{(9/8)^2 - (3/8)^2} = \sqrt{\dfrac{81}{64} - \dfrac{9}{64}} = \sqrt{\dfrac{72}{64}} = \dfrac{3\sqrt{2}}{4}$

$$\frac{(x-9/8)^2}{(3/8)^2} - \frac{(y-0)^2}{\left(3\sqrt{2}/4\right)^2} = 1$$

Another strategy to convert this to a rectangular equation is to rearrange the equation using cross multiplication, then square both sides before using the transformation equations.

$$r = \frac{3}{1+3\cos q}$$

$$r\left(1+3\cos q\right)=3$$

$$r+3r\cos q=3$$

$$r=3-3r\cos q$$

$$r^2=(3-3r\cos q)^2=(3-3r\cos q)(3-3r\cos q)$$

$$r^2=9-18r\cos q+9r^2\cos^2 q$$

$$r^2=9-18r\cos q+9(r\cos q)^2$$

Now use $r^2 = x^2 + y^2$

and $x = r\cos\theta$ to convert the equation.

$$x^2+y^2=9-18x+9x^2$$

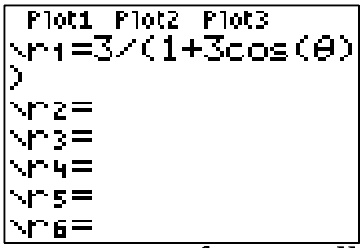
$$0=8x^2-18x-y^2+9$$

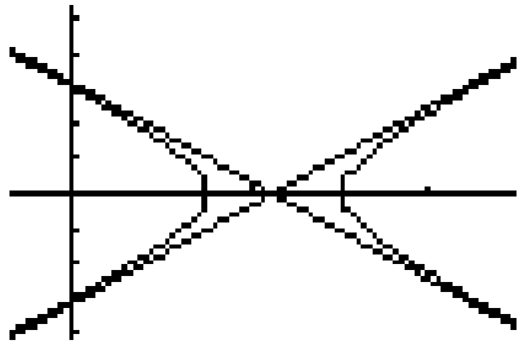
However, this form does not readily tell you which conic it is, what are the center, vertices, etc…

What is B^2-4AC?

0-4(8)(-1) = 32 >0 , therefore it is a hyperbola.

To make the polar graph on your calculator,
set mode to RADIAN and POL,
click on Y= and type in equation.





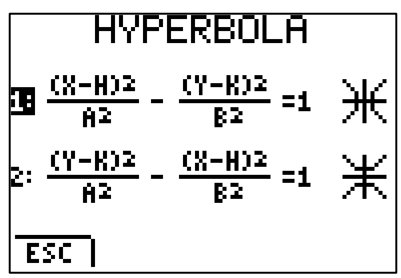
Choose Zoom:Fit. If you still can't see it well, choose Zoom: Zoom in, put the curser at the center and click on ENTER.

To graph the rectangular version of this hyperbola:

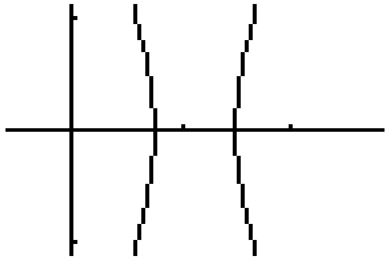
$$\frac{(x-9/8)^2}{(3/8)^2}-\frac{(y-0)^2}{\left(3\sqrt{2}/4\right)^2}=1$$

Set MODE to FUNCTION, click on APPS, and choose CONICS. Choose 3:HYPERBOLA. Since this hyperbola's major axis is the x-axis, choose option 1.

Set
A = 3/8
B = $3\sqrt{(2)}/4$
H = 9/8
K = 0



Then click on GRAPH.



6.7 Parametric Equations

Let x = f(t) and y = g(t), where f and g are two functions whose comon domain is some interval I. The collection of points defined by ***(x,y) = (f(t) , g(t))*** is called a plane curve. The equations x = f(t) and y = g(t) are called parametric equations of the curve. The variable t is called the parameter.

Parametric equations are particular useful in describing moving movement along a curve.

Example 1. Graph the curve defined by the parametric equations

$$x = 3t^2, \quad y = 2t, \text{ where } -2 \le t \le 2$$

To graph this on the x-y plane, solve for t in terms of x or y and substitute that expression for t in the other equation.
Let's use y = 2t to solve for t. t = y/2
Substitute y/2 for t in $x = 3t^2$.

$$x = 3\left(\frac{y}{2}\right)^2 = \frac{3}{4}y^2$$

$$y^2 = \frac{4}{3}x = 4\left(\frac{1}{3}\right)x$$

From our review of conic sections, this is a parabola pointing out in the direction of the positive x-axis with focus at (1/3, 0).

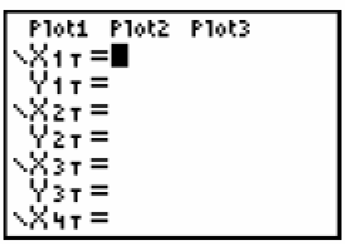
Setting Parametric Graphing Mode

To display the mode screen, press [MODE]. To graph parametric equations, you must select **Par** graphing mode before you enter window variables and before you enter the components of parametric equations.

1. Press [MODE]. Press [▼] [▼] [▼] [►] [ENTER] to select **Par** mode. Press [▼] [▼] [►] [ENTER] to select **Simul** for simultaneous graphing of all three parametric equations in this example.

Displaying the Parametric Y= Editor

After selecting **Par** graphing mode, press [Y=] to display the parametric **Y=** editor.



In this editor, you can display and enter both the **X** and **Y** components of up to six equations, X_{1T} and Y_{1T} through X_{6T} and Y_{6T}. Each is defined in terms of the independent variable **T**. A common application of parametric graphs is graphing equations over time.

Selecting a Graph Style

The icons to the left of X_{1T} through X_{6T} represent the graph style of each parametric equation (Chapter 3). The default in **Par** mode is ⋱ (line), which connects plotted points. Line, ⋱ (thick), ⋱ (path), ⋱ (animate), and ⋱ (dot) styles are available for parametric graphing.

Setting Window Variables

To display the window variable values, press [WINDOW]. These variables define the viewing window. The values below are defaults for **Par** graphing in **Radian** angle mode.

Tmin=0	Smallest **T** value to evaluate
Tmax=6.2831853...	Largest **T** value to evaluate (2π)
Tstep=.1308996...	**T** value increment (π/24)
Xmin=-10	Smallest **X** value to be displayed
Xmax=10	Largest **X** value to be displayed
Xscl=1	Spacing between the **X** tick marks
Ymin=-10	Smallest **Y** value to be displayed
Ymax=10	Largest **Y** value to be displayed
Yscl=1	Spacing between the **Y** tick marks

Notice that in our example Tmin = -2.

Note: To ensure that sufficient points are plotted, you may want to change the **T** window variables.

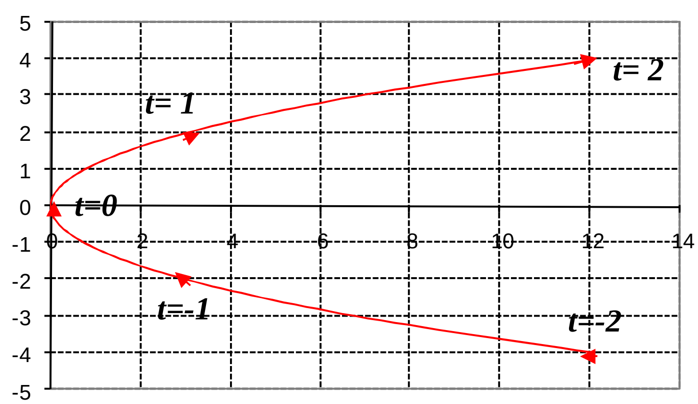
For our example let's set Tmin = -2, Tmax = 2, Tstep = 0.1 Xmin = 0, Xscl=1,Ymin = -5, and Ymax = 5.

The smaller the Tstep, the more points the graphing utility will plot.

To create a Table, go to 2nd TBLSET and assign TblStart to -2 and ΔTbl = 1

This will create a table that you can use to hand plot the parametric equations.

T	$X1_T$	$Y1_T$
-2	12	-4
-1	3	-2
0	0	0
1	3	2
2	12	4



Notice that your calculator will give you additional values for t but our example has t limited between the values of -2 and 2.

Example 3

Find the rectangular equation of the curve whose parametric equations are

$x = a \cos t$ and $y = \boldsymbol{a} \sin t$

where $a > 0$ is a costant.

Solution: The presence of sines and cosines in the parametric equations suggest that we use a Pyathagorean identity (recall $\sin^2\theta+\cos^2\theta=1$)

Rearranging the equations for x and y gives:

$x/\boldsymbol{a} = \cos t$ and $y/\boldsymbol{a} = \sin t$

and since $\sin^2 t+\cos^2 t = 1$, $\boldsymbol{(x/a)^2 + (y/a)^2} = 1$

Multiplying both sides by $\boldsymbol{a^2}$ gives

$\boldsymbol{x^2 + y^2 = a^2}$ is a circle with center (0,0) and radius $\boldsymbol{a}$.

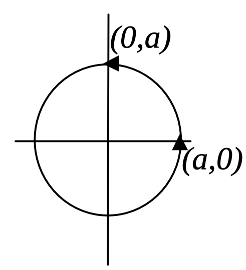
At $\mathbf{t = 0}$,

$(x(t), y(t))=(a\cos 0, a\sin 0) = (\boldsymbol{a},0)$.

At $t = \pi/2$,

$(x(t), y(t))=(\boldsymbol{a}\cos \pi/2, \boldsymbol{a}\sin \pi/2) = (0,\boldsymbol{a})$.

So the movement goes in counterclockwise direction.



Example 5 Projectile Motion

The parametric equations of the path of a projectile fired at an inclination of θ to the horizontal, with initial speed v_0, from a hieght *h* above the horizontal are:

$$x = (v_0 \cos q)t$$

$$y = -\frac{1}{2}gt^2 + (v_0 \sin q)t + h$$

where t is the time and g is the constant acceleration due to gravity (approximately 32ft/sec^2 or 9.8m/sec^2)

p.474 Example 5 Suppose Jim hit a golf ball with initial velocity of 150 ft/sec at an angle of 30° to the horizontal.

a) Find parametric equations that describe the position of the ball as a function of time.

$$x = (150\cos 30°)t = \left(150\frac{\sqrt{3}}{2}\right)t = 75\sqrt{3}t$$

$$y = -\frac{1}{2}(32)t^2 + (150\sin 30°)t + 0 = -16t^2 + (150(.5))t = -16t^2 + 75t$$

b) How long is the ball in the air?

-At what time t does y reach 0 again?

$y = -16t^2+75t=0$

$t(-16t+75) = 0$

t=0 (initial point) and t = 75/16 sec = 4.6875 sec

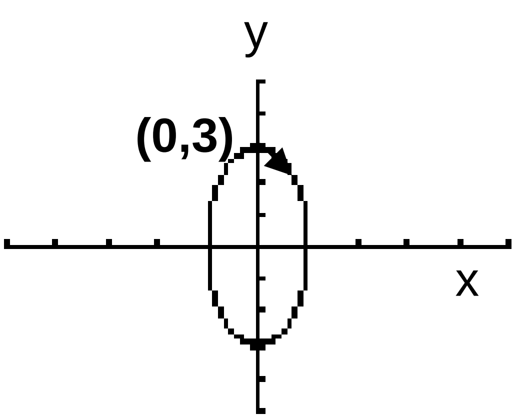
Example 8

Find the parametric equations for the ellipse

$$x^2 + \frac{y^2}{9} = 1$$

Where the parameter t is time (in seconds) and

a) The motion around the ellipse is clockwise, begins at the point (0,3), and requires 1 second for a complete revolution.



a) Let's look at the pattern of the *x*'s, then the *y*'s.

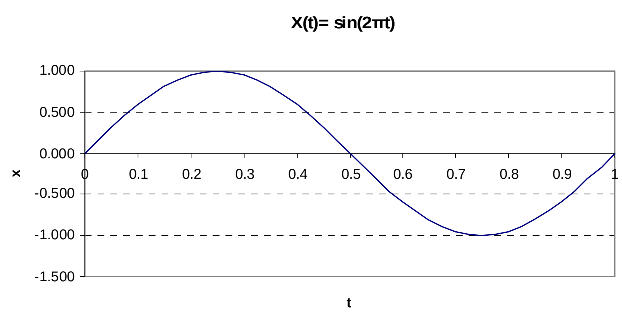
X:

x begins at 0, moves positively to 1, then negatively back to 0, then continues negatively to -1, the positively back to 0. This whole cycle takes place between t=0 and t=1. This should remind you of the sine curve with amplitude=1 and period=1.

To find the sine curve equation X(t), we set X(t) = Asin(ωt)

A = 1, Period = 2π/ω = 1. So ω= 2π

Therefore X(t) = sin(2π t)

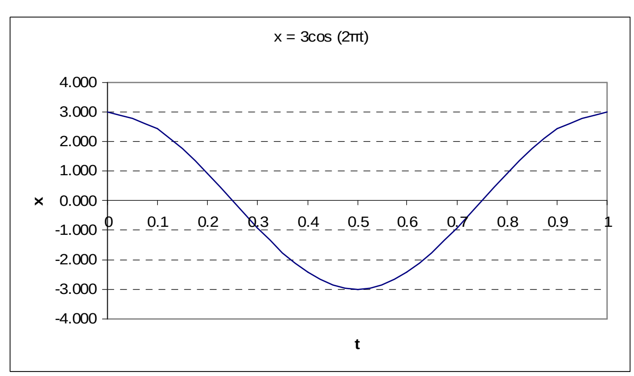


The pattern of the y's is as follows: y begins at 3 when t=0, moves negatively toward 0, then continues negatively towards -3, then positively towards 0, and completes the cycle at y=3 when t=1. This should remind you of the cosine curve with amplitude 3 and period 1.

To find the cosine curve equation X(t), we set X(t) = Acos(ωt)

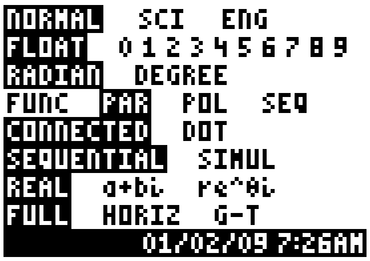
A = 3, Period = 2π/ω = 1. So ω= 2π

Therefore X(t) = 3cos(2π t)

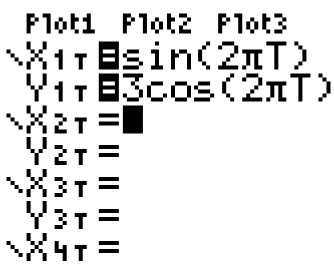


Checking if your parametric equations form an ellipse on your graphing calculator.

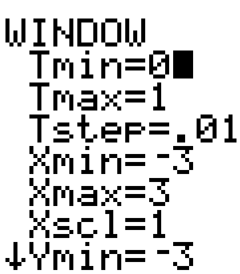
Go to MODE and set third line to RADIAN and fourth line to PAR *(parametric).*

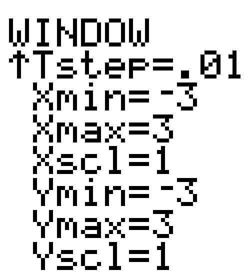


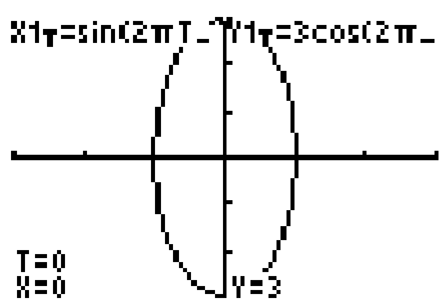
Go to Y= and input your parametric equations.



Go to WINDOW and set Tmin 0 to Tmax to 1 (since $0 \le t \le 1$). In order to get a more refined ellipse, make the Tstep small, such as .01. Set Xmin = -3, Xmax = 3, Ymin = -3, Ymax = 3. Then go to GRAPH. You can hit TRACE trace the parametric function's movement as t progresses.







PROJECT 5

p.486 #1,5,9,12,13
p.413 #1-21 EOO
p.332 #1,3,4,5,7,13
p.286 #1,3,5,17,21,25
p.215 #1-31 EOO
p.111 #1-15 ODD