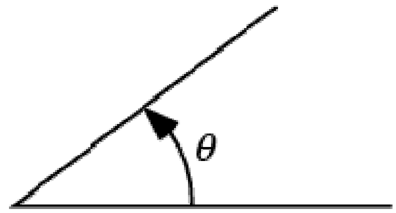
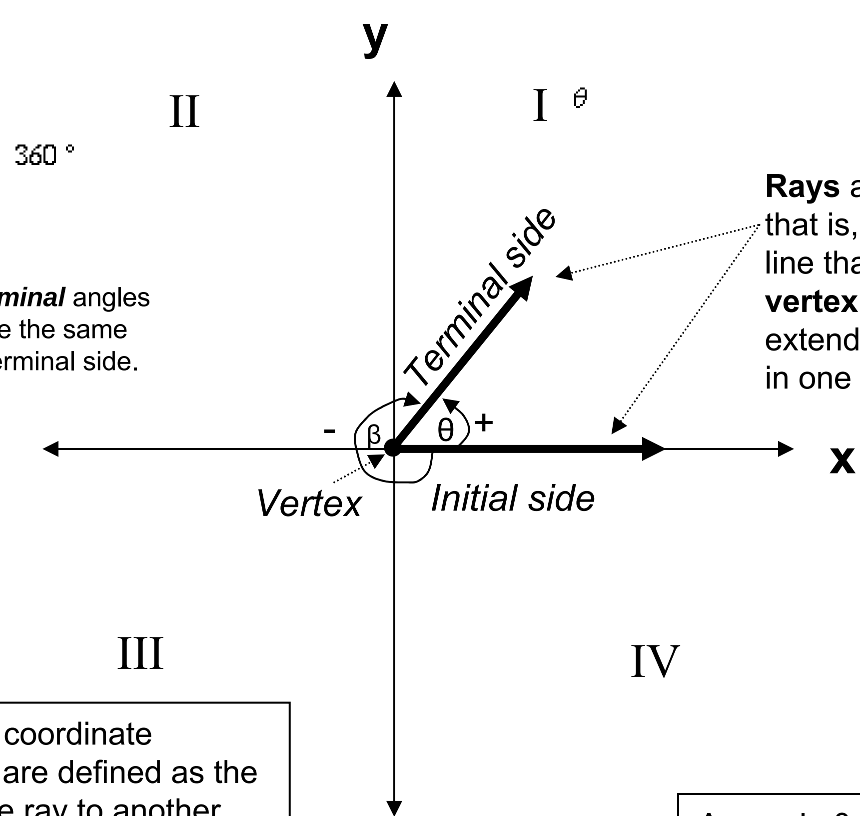
2.1 ANGLES AND THEIR MEASURE



Given two intersecting lines or line segments, the amount of rotation about the point of intersection (the vertex) required to bring one into correspondence with the other is called the angle   between them. Angles are usually measured in degrees (denoted °   ), radians (denoted rad, or without a unit), or sometimes gradians (denoted grad).

β and θ are ***coterminal*** angles because they have the same initial and same terminal side.



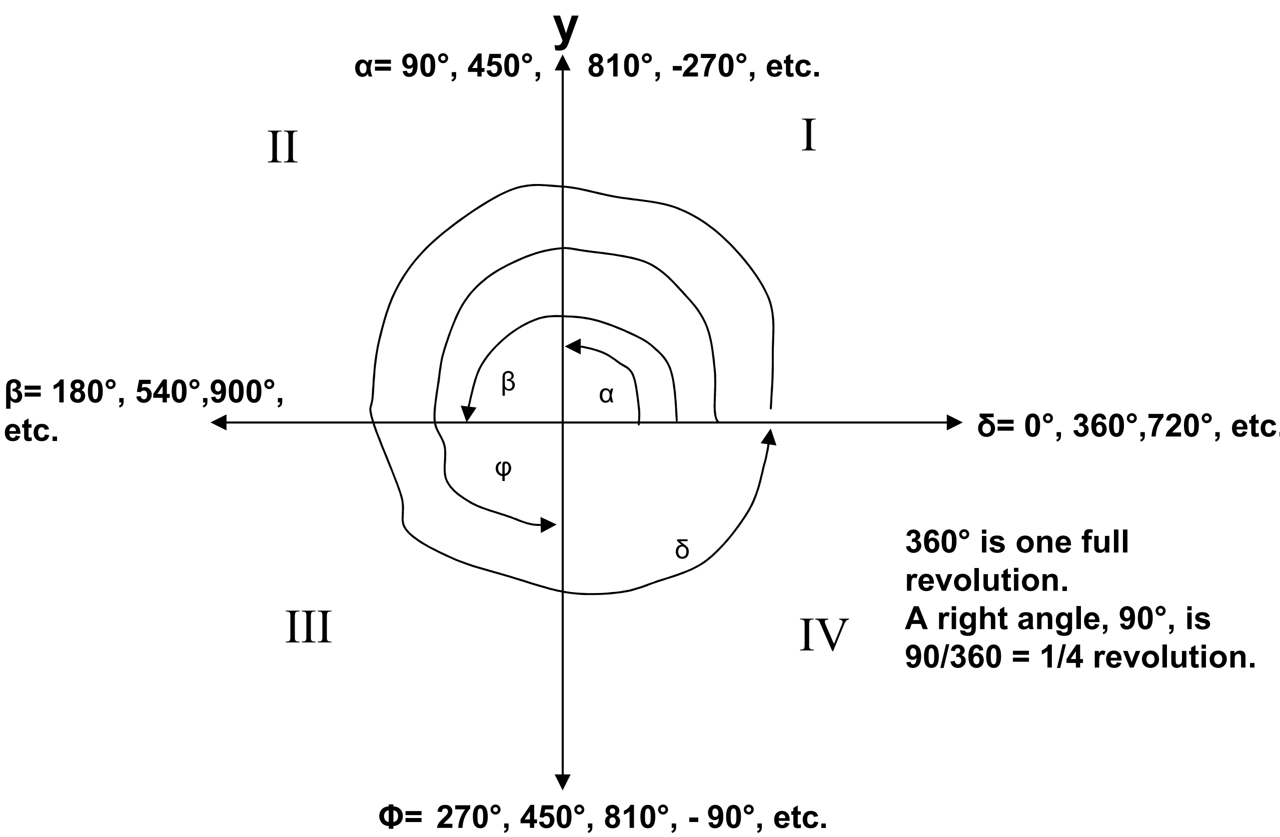
Rays are half-lines, that is, a portion of a line that starts a **vertex** point and extends indefinitely in one direction.

In a rectangular coordinate system, **angles** are defined as the rotation from one ray to another. The ray at the starting position is called the **initial side** and the ray at then ending position is the **terminal side**. Rotation in a counterclockwise direction is **positive** and rotation in a clockwise direction is **negative**.

An angle θ is in **standard position** if its vertex is at the origin and its initial side coincides with the positive x-axis.

Quadrantal Angles

One full rotation in these three measures corresponds to **360°,** 2π radians, or 400 gradians. Half a full rotation is 180° and is called a straight angle, and a quarter of a full rotation is 90° and is called a right angle. An angle less than a right angle is called an acute angle, and an angle greater than a right angle is called an obtuse angle.



360° is one full revolution.
A right angle, 90°, is 90/360 = 1/4 revolution.

Converting from Degrees, Minutes, Seconds (D°, M',S") to Decimal Form

The use of degrees to measure angles harks back to the Babylonians, whose sexagesimal number system was based on the number 60. likely arises from the Babylonian year , which was composed of 360 days (12 months of 30 days each). The degree is further divided into 60 arc minutes (denoted **'**), and an arc minute into 60 arc seconds(denoted **"**). A more natural measure of an angle is the radian. It has the property that the arc length around a circle is simply given by the radian angle measure times the circle radius. The radian is also the most useful angle measure in calculus. Gradians are sometimes used in surveying (they have the nice property that a right angle is exactly 100 gradians), but are encountered infrequently, if at all, in mathematics.

1 counterclockwise revolution = 360°
1°=60' = 60 minutes
1 minute = 1'= 60" = 60 seconds

Example 2 on p.101
Convert 50°6'21" to decimal in degrees.
1°=60' which means 1' = 1°/60
1'= 60" so 1" = 1'/60 = (1°/60)/60 = 1°/3600
50°6'21" = 50 + 6'*1°/60 + 21"*1°/3600 = <u>50.105833°</u>

Convert 21.256° to (D°, M',S") in degrees.
Start with the decimal part, .256.
.256 ° = .256*60'/ 1° = 15.36'
Take the decimal part of 15.36 and convert to seconds.
.36 ' * 60"/1'= 21.6 ≈ 22"
Thus,
21.256°=21°+15'+21.6" ≈ <u>21°15'22"</u>

Converting from Degrees, Minutes, Seconds (D°, M’,S”) to Decimal Form Using TI-83 or TI-83 plus

Convert 50°6’21” to decimal in degrees.

5 0	2nd	[ANGLE]	1	6	2nd	[ANGLE]	2	2 1	ALPHA	“	ENTER

↑ "1": Selects Degree notation

↑ "2": Selects Minute notation

↑ "“": Selects Second notation

ANGLE Operations

ANGLE Menu

To display the **ANGLE** menu, press [2nd] [ANGLE]. The **ANGLE** menu displays angle indicators and instructions. The **Radian**/**Degree** mode setting affects the TI-83 Plus’s interpretation of **ANGLE** menu entries.

ANGLE	
1: °	Degree notation
2: '	DMS minute notation
3: r	Radian notation
4: ▶DMS	Displays as degree/minute/second
5: R▶Pr(	Returns **r**, given **X** and **Y**
6: R▶Pθ(	Returns **θ**, given **X** and **Y**
7: P▶Rx(	Returns **x**, given **R** and **θ**
8: P▶Ry(	Returns **y**, given **R** and **θ**

To convert from decimal to DMS, just type the number in decimal form, then press 2nd [ANGLE] 4 to see it in DMS form.

Entry Notation

DMS (degrees/minutes/seconds) entry notation comprises the degree symbol (°), the minute symbol ('), and the second symbol ("). *degrees* must be a real number; *minutes* and *seconds* must be real numbers ≥ 0.

degrees°*minutes*'*seconds*"

▶DMS

▶DMS (degree/minute/second) displays *answer* in DMS format. The mode setting must be **Degree** for *answer* to be interpreted as degrees, minutes, and seconds. **▶DMS** is valid only at the end of a line.

answer▶**DMS**

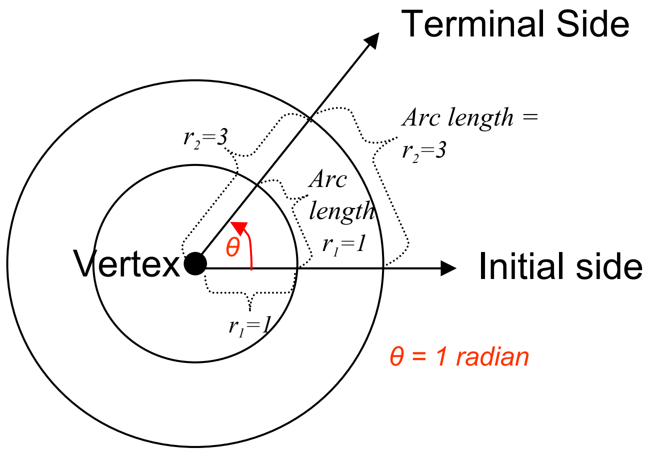
```
54°32'30"*2
        109.0833333
Ans▶DMS
           109°5'0"
```

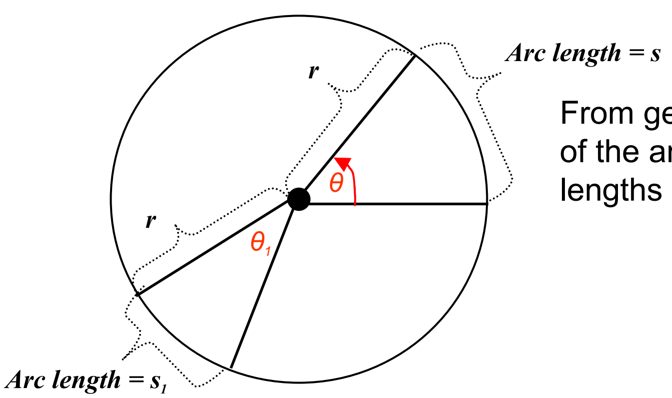
Central Angles and Arc Length

A central angle is an angle whose vertex is at the center of a circle. The rays of a central angle subtend (intersect) an arc on the circle. [The central angle is between 0 and 360 degrees].

1 radian is defined as a central angle of a circle with radius r that has an arc length of r.

If the circumference of a circle is $2\pi r$, how many radians are there in 1 full revolution of the circle?_____





From geometry, we know that the ratio of the measures of the angles equals the ratio of the corresponding lengths of the arcs subtended by these angles.

$$\frac{\theta}{\theta_1} = \frac{s}{s_1}$$

If θ_1 = 1 radian, then the arc length s_1 = r, so

$$\frac{\theta}{1} = \frac{s}{r} \text{ and cross-multiplying gives } s = r\theta \text{ for any } \theta \text{ in the circle}$$

ARC LENGTH

For a circle of radius r, a central angle of θ radians subtends an arc whose length s is

$$s = r\theta$$

Now try #37 on p.109

How do you find arc length if the angle is given in degrees?

Recall that the circumference of a circle is $2\pi r$.

The arc length of a full revolution (θ_{full}) is $2\pi r = r\theta_{full}$

Therefore $\theta_{full} = 2\pi$ radians

1 full revolution = 360° = 2π radians

½ revolution = 180° = π radians

¼ revolution = 90° = π/2 radians

We use this fact to convert from degrees to radians.
1 radian = 180° / π
1 degree = π radians/180

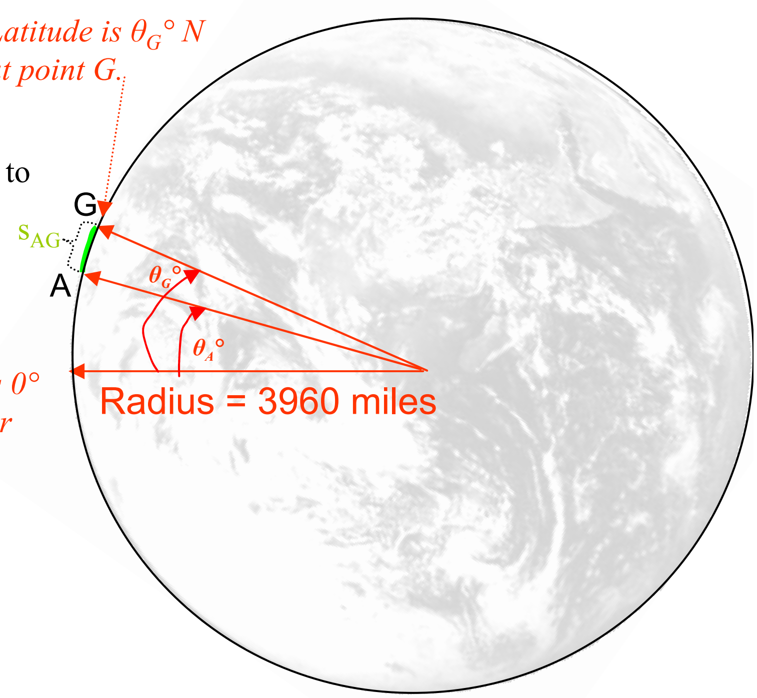
Example 4 (a) on p. 104
Convert 60 ° ***to radians.***

60 ° * π radians/180 = 60 π /180 radians = π/3 radians
Now you do #13 on p.109

Example 5(c) on p.105
Convert -3π/4 radians to degrees.
(-3π/4)*180° / π = -135°
Now you do #25 on p.109

Example 6 on p.105
Latitude of a location is the angle formed by a ray drawn from the center of Earth to the Equator and a ray drawn from the center of Earth to the location.
Glasgow, Montana is due north Albuquerque, New Mexico. Glasgow's latitude is 48°9' N and Albuquerque's latitude is 35°5' N. Find the distance between these 2 cities.

s_{AG} is the arc length of the central angle, θ_{AG}, subtended by the ray from Earth's center to Glasgow and the ray from Earth's center to Albuquerque. That is the distance from Glasgow to Albuquerque.



Step 1:

$\theta_{AG} = \theta_G - \theta_A$ = 48°9' - 35°5'= 13°4'

Step 2:
Convert from DMS to decimal degrees
13°4'=13 + 4'/(1°/60') ≈ 13.0667°

Step 3:
Convert from decimal degrees to radians.
13.0667°*(π radians/180) ≈ 0.228 radians.

Step 4:
Calculate arc length s_{AG} .
$s_{AG} = R_{Earth} * \theta_{AG\ radians}$ = *3960** 0.228 ≈ **903 miles**

Now do #91 on p.110

Common Degree to Radian Conversions

Degrees	Radians	
0	0	
30	π/6	
45	π/4	*
60	π/3	
90	π/2	
120	2π/3	
135	3π/4	
150	5π/6	
180	π	*
210	7π/6	
225	5π/4	
240	4π/3	
270	3π/2	*
300	5π/3	
315	7π/4	
330	11π/6	
360	2π	*

*** Memorize these**

Area of a Sector

From geometry, we know that the ratio of the measures of the angles equals the ratio of the corresponding areas of the sectors formed by these angles. That is,

$$\frac{\theta}{\theta_1} = \frac{A}{A_1}$$

If θ_1 = 2π radians, then A_1 = area of circle.

$$\frac{\theta}{2\pi} = \frac{A}{\pi r^2}$$

$$\theta = 2\pi \bullet \frac{A}{\pi r^2} = \frac{2A}{r^2}$$

Multiply both sides by $\frac{r^2}{2}$ to solve for A.

Area of a sector formed by a central angle of θ radians is

$$A = \frac{1}{2} r^2 \theta$$

Now you do #45 on p.109

Linear and Angular Speed

Linear speed = distance traveled around a circle (s) divided by the elapsed time of travel (t).

velocity = v = s/t

Angular speed = the central angle swept out in time (θ), divided by the elapsed time, (t).

Angular speed = ω = θ/t

where angular speed is in radians per unit time.

Example:

An engine is revving at 900 RPM. If you are given RPM instead of angular speed, you can convert to angular speed by using that fact that 1 revolution = 2π radians

$$900\frac{revolutions}{minute} = 900\frac{\text{revolutions}}{\text{minute}} \bullet \frac{2\pi\text{ radians}}{1\text{ revolution}} = 1800\pi\frac{\text{radians}}{\text{minute}}$$

Linear speed = v = s/t = (rθ)/t = r(θ/t)= rω

where ω is measured in radians per unit time.

Example 8 on p.108

A child is spinning a rock at the end of a 2-foot rope at a rate of 180 revolutions per minute. Find the liear speed of the rock when it releases.

$$180\frac{revolutions}{minute} = 180\frac{\text{revolutions}}{\text{minute}} \bullet \frac{2\pi\text{ radians}}{1\text{ revolution}} = 360\pi\frac{\text{radians}}{\text{minute}}$$

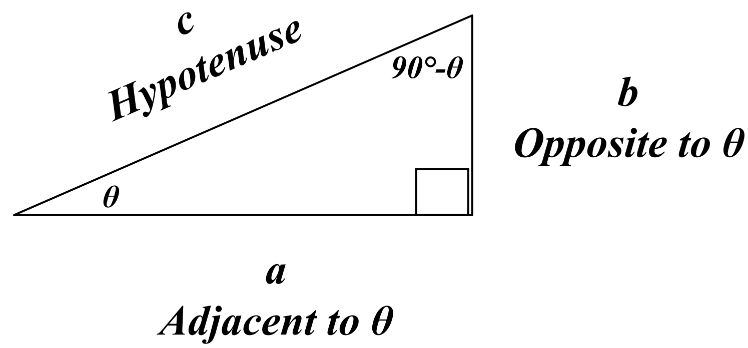
Linear speed v = rw = (2ft) * (360π radians/minute) = 720π feet/min ≈ 2262 feet/min

$$2262\frac{\text{feet}}{\text{min}} * \left(\frac{1\text{ mile}}{5280\text{ feet}}\right) * \left(\frac{60\text{ min}}{1\text{hour}}\right) \approx 25.7\text{ mph}$$

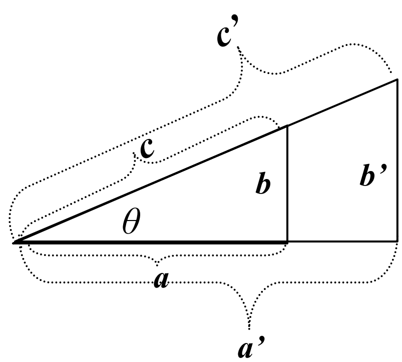
Now you do # 87 on p.110

Right Triangle Trigonometry

*θ is an **acute** angle because it is less than 90 degrees.*



From geometry, we know that the ratios of the sides of similar triangles are equal so:



$$\frac{b}{c}=\frac{b'}{c'},\ \frac{a}{c}=\frac{a'}{c'},\ \frac{b}{a}=\frac{b'}{a'}$$

$$\frac{c}{b}=\frac{c'}{b'},\ \frac{c}{a}=\frac{c'}{a'},\ \frac{a}{b}=\frac{a'}{b'}$$

These ratios are the same for any right triangle with acute angle *θ*. They are called the trigonometric functions of acute angles.

FUNCTION NAME	ABBREV.	VALUE
Sine of *θ*	sin(*θ*)	*b/c=**o**pposite/**h**ypotenuse*
Cosine of *θ*	cos(*θ*)	*a/c=**a**djacent/**h**ypotenuse*
Tangent of *θ*	tan(*θ*)	*b/a=**o**pposite/**a**djacent*
Cosecant of *θ*	csc(*θ*)	*c/b=hypotenuse/opposite*
Secant of *θ*	sec(*θ*)	*c/a=hypotenuse/adjacent*
Cotangent of *θ*	cot(*θ*)	*a/b=adjacent/opposite*

Notice these functions are the reciprocals of sine, cosine, & tangent, respectively.

Remember SOH-CAH-TOA!

In other words:

csc(*θ*)=*1*/sin(*θ*), sec(*θ*)=*1*/cos(*θ*), cot(*θ*)=*1*/tan(*θ*)

Finding Trig Functions

Example 2 on p.114 Finding the Value of the Remaining Trig Functions, given sin and cos.

Given $\sin(\theta) = \frac{\sqrt{5}}{5}$, and $\cos(\theta) = \frac{2\sqrt{5}}{5}$, find the remaining trig functions of θ

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{\sqrt{5}}{5}}{\frac{2\sqrt{5}}{5}} = \frac{1}{2}$$

$$\sec(\theta) = 1/\cos(\theta) = \frac{5}{2\sqrt{5}} \quad = \frac{\sqrt{5}}{2}$$

$$\csc(\theta) = 1/\sin(\theta) = \frac{5}{\sqrt{5}} \quad = \sqrt{5}$$

NOW YOU DO #11 on p121

Fundamental Identities of Trigonometry

We can use the Pythagorean Theorem to derive the fundamental identities of trigonometry.

We know $a^2 + b^2 = c^2$, right? Let's rearrange some terms and divide each side by c^2

$$b^2 + a^2 = c^2$$

$$\frac{b^2}{c^2} + \frac{a^2}{c^2} = \frac{c^2}{c^2} = 1$$

$$\left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = 1$$

$$(\sin(\theta))^2 + (\cos(\theta))^2 = 1$$

which can also be written as follows:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

Now we can get another identity by dividing both sides of this equation by $\cos^2(\theta)$

$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

Similarly, you had divided each side of that equation by $\sin^2(\theta)$:

$$\frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

Example 3 on p.115

Fundamental Identities

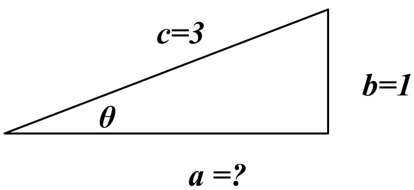
$\tan(\theta) = \sin(\theta)/\cos(\theta)$ $\cot(\theta) = \cos(\theta)/\sin(\theta)$

$\csc(\theta) = 1/\sin(\theta)$ $\sec(\theta) = 1/\cos(\theta)$ $\cot(\theta) = \cos(\theta)/\sin(\theta)$

$\sin^2(\theta) + \cos^2(\theta) = 1$ $\tan^2(\theta) + 1 = \sec^2(\theta)$ $1 + \cot^2(\theta) = \csc^2(\theta)$

Finding the exact value of the trig functions, given one.

Example 4
Given that sin(θ)= 1/3 and θ is an acute angle, find the exact value of each of the remaining five trigonometric functions of θ.
Remember sin(θ)= 1/3 = opposite/hypotenuse so let's make a right triangle with the opposite of θ to be 1 and the hypotenuse to be 3.
We can use the Pythagorean Theorem to find the adjacent side, a.



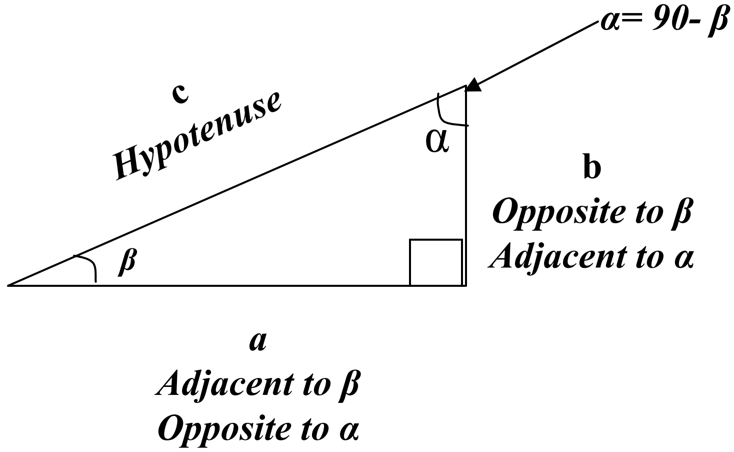
$$a^2 = 3^2 - 1^2 = 9 - 1 = 8$$

$$a = \sqrt{8} = \sqrt{4 \bullet 2} = 2\sqrt{2}$$

Now that you know all three sides of the triangle, just use the definitions to find the exact values of the trig functions.

Trig Function	Definition	Exact Value
sin(*θ*)	*b/c=**o**pposite/**h**ypotenuse*	*b/c=1/3*
cos(*θ*)	*a/c=**a**djacent/**h**ypotenuse*	*a/c=* $\frac{2\sqrt{2}}{3}$
tan(*θ*)	*b/a=**o**pposite/**a**djacent*	*b/a=* $\frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$
csc(*θ*)	*c/b=hypotenuse/opposite*	*c/b=3/1 = 3*
sec(*θ*)	*c/a=hypotenuse/adjacent*	*c/a=* $\frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$
cot(*θ*)	*a/b=adjacent/opposite*	*a/b=* $\frac{2\sqrt{2}}{1} = 2\sqrt{2}$

Complementary Angles; Cofunctions



α and β are complementary angle because their sum is a right angle, 90 °.

Complementary Angle Theorem

Cofunctions of complementary angles are equal. Cofunctions are trigonometric functions that share the same angles of a right triangle.

sin(*β*)=opposite to β / hypotenuse = adjacent to α/hypotenuse = cos(α)=cos(*90- β*)

cos(*β*)=adjacent to β / hypotenuse = opposite to α/hypotenuse = sin(α)=sin(*90- β*)

tan(*β*)=opposite to β / adjacent to α = adjacent to α/ opposite to β = cot(α)= cot(*90- β*)

<u>Likewise, the reciprocal of these properties is true.</u>

csc(*β*)= hypotenuse / opposite to β / = hypotenuse /adjacent to α= sec(α)=sec(*90- β*)

sec(*β*)= hypotenuse /adjacent to β= hypotenuse/opposite to α = csc(α)= csc(*90- β*)

cot(*β*)= adjacent to α /opposite to β = opposite to β / adjacent to α = tan(*90- β*)

HOMEWORK

Homework

p.109-110 #15, 23, 27, 39, 41, 47, 51, 53, 57, 69,75,81,85,91

p. 121 #3, 13,17, 23,37, 47, 53, 57